The Derivations of Evolution Algebras under a Given Conditions

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ABSTRACT

In the theory of non-associative algebras, particularly in genetic algebras, Lie algebra. The derivations of a given algebra are one of the important tools for studying its structure. This work investigates the derivations of n-dimensional complex evolution algebras based on the rank of appropriate matrices. The spaces of derivations of evolution algebras under some possible conditions with matrices of rank n-2 are investigated.

Keywords: *Derivation, Evolution Algebras, Rank of Matrix.*

I. INTRODUCTION AND PRELIMINARIES

Evolution algebras have been introduced in [8], here the author indicated to be connections between evolution algebras with other mathematical fields, genetic and physics. Evolution algebras are not considered to be of any well-known classes of non-associative algebras, as Lie, alternative and Jordan algebras since they are not defined by identities. In between algebras and dynamical systems, evolution algebras are presented as a new field which connected with both previous mentioned fields and they could be defined algebraically, their structure has a table of multiplication, which satisfies the conditions of commutative Banach algebra as non-associative Banach algebra in general; dynamically, they represent discrete dynamical systems [10]–[14].

It is known that multiplication is defined in terms of derivation, which means that derivations in genetic algebra are of great importance. Moreover, many explanations for the derivation of genetics have been provided [5, 6]. Since then, the derivation of certain algebra is one of the most important tools that need to be verified in its structure. In addition, [2-4, 13] investigated the derivations of a given algebras.

Let E be a vector space over a field K with a basis $\{e_1, e_2, \ldots\}$ and a multiplication rule ⊔ such that $e_i \cdot e_j = \begin{cases} 0, & i \neq j \\ \sum_m a_{ik} e_m, & i = j \end{cases}$ then E is called evolution algebras. The basis is called a natural basis and the matrix $A = \left(a_{ij}\right)_{i,j=1}^{n}$ is denoted the matrix of the structural constants of the finite-dimensional evolution algebra E. Drawing on the definition of evolution algebras, the evolution algebras are commutative, this fact lead us to judge it is flexible. The rank of the matrix for finite-dimensional evolution algebra does not depend on choice of natural basis since $rankA = \dim(E \cdot E)$.

The derivation of evolution algebra E is defined as usual, i.e., a linear operator $d : E \to E$ is called a derivation if $d(u \cdot v) = d(u) \cdot v + u \cdot d(v)$ for all $u, v \in E$.

The space $Der(E) = \{d \in End(E) \mid a_{kj} d_{ij} + a_{ki} d_{ij} = 0, \; for\; i \neq j; 2a_{ji} d_{ii} = \sum_{k=1}^{n} a_{ki} d_{jk}\}$ of all derivations for any algebra is Lie algebra with the commutator multiplication. The basic properties and some classes of evolution algebras were studied as well in [1, 8, 13]. In [7, 9], the space of a derivation of evolution algebra with non-singular matrices and with matrices of rank $n-1$ have been described. This work will be described the space of derivations of evolution algebra with matrix of rank $n-2$ under a certain possible conditions

II. DESCRIPTION OF DERIVATION OF EVOLUTION ALGEBRA

According to definition of derivations of evolution algebra, it easy to see that $d(e_i)e_j + e_i d(e_j) = 0$ and $d(e_i^2) = 2d(e_i^2)$ for all $1 \leq i \neq j \leq n$. Now, let $d(e_k) = \sum_{i=1}^{n} d_{ki} e_i$. Then, the following are obtained:

$$d_{ij}\left(e_j^2\right) + d_{ji}\left(e_i^2\right) = 0 \quad (1)$$

$$d\left(e_i^2\right) = 2d_{ii}\left(e_i^2\right) \quad (2)$$

for all $1 \leq i \neq j \leq n$

Plainly, $e_1^2,\ldots,e_{n-2}^2$ should be linearly independent, this is because of $rank\ A = n-2$. As a result of performing a suitable basis permutation e_i^2 such that $i \in \{n, n-1\}$ can be written by $e_i^2 = \sum_{k=1}^{n-2} b_k \left(e_k^2\right)$, $i \in \{n-1, n\}$, $b_1,\ldots,b_{n-1} \in \square$.

Now, relying on (2), one can show simply that $2d_{ii}$ is an eigenvalues of d for all $1 \le i \le n-2$ as an outcome of this $spec(d) \supseteq \{2d_{11}, 2d_{22}, \ldots, 2d_{n-2\,n-2}\}$. In addition, (1) points out that $d_{ij} = d_{ji} = 0$ for all $1 \le i \ne j \le n-2$ if i is replaced by n and $n-1$ in (1) separately afterwards the following is achieved:

$$d_{mj}\left(e_j^2\right) + d_{jm}\left(e_n^2\right) = 0,\ m\{n, n-1\} \qquad (3)$$

$$\left(d_{n-1j} + d_{jn-1}b_j\right)\left(e_j^2\right) + \sum_{k=1,k\ne j}^{n-2} d_{jn-1}b_k\left(e_k^2\right) = 0 \qquad (4)$$

$$\left(d_{nj} + d_{jn}b_j\right)\left(e_j^2\right) + \sum_{k=1,k\ne j}^{n-2} d_{jn}b_k\left(e_k^2\right) = 0 \qquad (5)$$

Consequently, (4) and (5) make that $d_{jn}b_j = 0$, $d_{nj} + d_{jn}b_j = 0$, $d_{jn-1}b_k = 0$ and $d_{n-1j} + d_{jn-1}b_j = 0$ for all $1 \le k \ne j \le n-2$.

In this work, tow possible different values of b_k will be investigated. To begin, consider the following:

$$\begin{pmatrix}
d_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & d_{1n-1} & d_{1n} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 2d_{11} & \cdots & 0 & 0 & 0 & \cdots & 0 & d_{s+1n-1} & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 2^{(k-s)2}d_{11} & 0 & 0 & \cdots & 0 & d_{k-n-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{11} & 0 & \cdots & 0 & d_{kn-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \frac{d_{n-1n-1}}{2^{(n-1)-(k+1)}} & \cdots & 0 & d_{k+1n-1} & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{d_{n-1n-1}}{2} & d_{n-2n-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & d_{n-1n-1} & 0 \\
-b_1d_{1n} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & d_{nn-1} & d_{11}
\end{pmatrix} \qquad (D_1)$$

$$\begin{pmatrix}
d_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{1n-1} & d_{1n} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & d_{s+1s+1} & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{s+1n-1} & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 2^{(m-(s+1))}d_{s+1s+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{mn-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 2d_{11} & \cdots & 0 & 0 & \cdots & 0 & d_{mnn-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 2^{(k-(m+1)+1)}d_{11} & 0 & \cdots & 0 & d_{kn-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \frac{d_{n-1n-1}}{2^{(n-1)-(k+1)}} & \cdots & 0 & d_{k+1n-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{d_{n-1n-1}}{2} & d_{n-2n-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{n-1n-1} & 0 \\
-bd_{1n} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{nn-1} & d_{11}
\end{pmatrix} \qquad (D_2)$$

$$\begin{pmatrix}
d_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{1n-1} & d_{1n} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & \frac{d_{11}}{2^{n-s-2}} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{s+1n-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & \frac{d_{11}}{2} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{k-1n-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{kn-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{d_{n-1n-1}}{2^{(n-1)-(q+1)}} & \cdots & 0 & d_{q+1n-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{d_{n-2n-1}}{2} & d_{n-1n-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{n-1n-1} & 0 \\
-bd_{1n} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{nn-1} & d_{11}
\end{pmatrix} (D_3)$$

$$\begin{pmatrix}
0 & \dots & d_{1n-1} & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \dots & d_{n-2n-1} & 0 \\
0 & \dots & 0 & 0
\end{pmatrix} \qquad (D_4)$$

Where $\sum_{k-1}^{n-2} a_{ik} d_{kn-1} = 0, 1 \le i \le n-2;$

$$\begin{pmatrix}
0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & \dots & 0 & \frac{d_{n-1n-1}}{2^{(n-1)-k-1}} & \dots & 0 & d_{k+1n-1} & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\
0 & \dots & 0 & 0 & \dots & \frac{d_{n-1n-1}}{2} & d_{n-2n-1} & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & d_{n-1n-1} & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & 0
\end{pmatrix} \qquad D_5$$

where $d_{i+1n-1} = \frac{a_{in-1}}{a_{ii+1}} \left(\frac{1}{2^{(n-1)-i-1}} - 1 \right) d_{n-1n-1}, k+1 \le i \le n-2,$ $1 \le k \le n-2$ and $d_{k+1n} \in \square$

Remark: *To avoid repetition the main different steps will be mentioned only.*

Theorem: *Let evolution algebra has a matrix* $A = \left(a_{ij}\right)_{1 \le i, j \le n}$ *in the natural basis* $e_1, \dots, e_n$ *such that* $e_{n-1}^2 = 0$ *and* $e_n^2 = b\left(e_1^2\right), b \ne 0$. *Then, derivation* d *of this evolution algebra is either zero or it is in one of the following forms up to basis permutation:*

(1) (D_1) *where* $d_{11} = \frac{\delta}{2^{(k-s)+1} - 11}$ *and* $\delta^2 = -bd_{1n}^2$.

(2) (D_2) *where* $d_{11} = \frac{\delta}{2^{k-(m+1)+2} - 1}, d_{s+1s+1} = \frac{1 - 2^{(k-(m+1)+1)}}{1 - 2^{m-(s+1)+1}} d_{11}$ *and* $\delta^2 = -bd_{1n}^2$.

(3) (D_3) *where* $d_{11} = \delta, \delta^2 = -bd_{1n}^2$.

(4) (D_4) *or* (D_5) *when* $d_{1n} = 0$.

Proof: From (1), we infer that $d_{n-1j} = 0$ for all $1 \le j \le n-2$, $d_{2n} = \dots = d_{n-2n} = 0$, $d_{n2} = \dots d_{nn-2} = 0$, $d_{n-1n} = 0$ and $d_{n1} = -bd_{1n}$. Let $i = n$ in (2). Then,

$$2bd_{11}\left(e_1^2\right) = bd\left(e_1^2\right) = d\left(b\left(e_1^2\right)\right)$$
$$= d\left(e_n^2\right) = 2d_{nn}\left(e_n^2\right) = 2d_{nn}b\left(e_1^2\right)$$

Hence, $d_{11} = d_{nn}$. Moreover, (2) implies

$$a_{i1}\left(d_{11}e_1 + d_{1n-1}e_{n-1} + d_{1n}e_n\right) + \sum_{j=2}^{n-2} a_{ij}\left(djjej + d_{jn-1}e_{n-1}\right)$$

$$+ a_{in}\left(-bd_{1n}e_1 + d_{nn-1}e_{n-1} + d_{11}e_n\right) = d\left(e_i^2\right) = 2d_{ii}e_i^2 = 2d_{ii}\sum_{j=1}^{n} a_{ij}e_j$$

Thus,

$$a_{i1}\left(2d_{ii} - d_{11}\right) = -a_{in}d_{1n}b \qquad (6)$$

$$a_{in}\left(2d_{ii} - d_{11}\right) = a_{i1}d_{1n} \qquad (7)$$

$$a_{ij}\left(2d_{ii} - d_{11}\right) = 0 \qquad (8)$$

$\sum_{j=1}^{n} a_{ij} d_{jn-1} e_{jn-1} = 2_{ain-1} d_{ii}$, for all $1 \le i \le n-2$.

To investigate the derivation of this case we are going to consider two cases that are $d_{1n} = 0$ and $d_{1n} \ne 0$.

Case 1: Let $d_{1n} \ne 0$. Then, $spec(d) = \{d_{22}, \dots d_{n-1n-1}, \alpha, \beta\}$ is easy to find where $\alpha = d_{11} + \delta$, $\beta = d_{11} - \delta$ and $\delta^2 = -bd_{1n}^2$. Obviously, $\alpha \ne \beta$. Let $\lambda \in spec(d)$ be such that $|\lambda| = \max\{|\alpha|, |\beta|, |d_{22}|, \dots, |d_{n-2n-2}|\}$. Clearly, $2d_{ii}$ is an eigenvalues for all $1 \le i \le n-2$ and therefore if $\lambda \in \{d_{22}, \dots, d_{n-2n-2}\}$ we immediately get 2λ is also an eigenvalues which contradicts to module maximality of λ. Therefore $\lambda = \alpha$, $\lambda = \beta$ or $\lambda = d_{n-1n-1}$. In addition, (6) and (8) follow that $a_{in} = 0$ if and only if $a_{in} = 0$.

Let $a_{i1} \ne 0 (a_{in} \ne 0)$. Then, by multiplying (6) by (7) derives $(2d_{ii} - d_{11})^2 = -bd_{1n}^2$ or $2d_{ii} = d_{11} \pm \delta$. Hence,

$$d_{ii} = \frac{1}{2}\alpha \text{ or } d_{ii} = \frac{1}{2}\beta \text{ for these } i. \quad (9)$$

Now, we are going to consider several cases.

Case 1.1: Let $\alpha\beta \ne 0$ such that $\alpha + \beta \ne 0$. Clearly, $\alpha + \beta = 2d_{11} \in spec(d)$ and $\alpha + \beta \notin \{\alpha, \beta, d_{n-1n-1}\}$. This presents that there exist $i1$ such that $d_{i1i1} = \alpha + \beta$ i.e. $2d_{i1i1} \in spec(d)$ which implies that $2d_{i1i1} = d_{i2i2}$ for some $i2$ or $2d_{i1i1} \in \{\alpha, \beta, d_{n-1n-1}\}$. If $2d_{i_1i_1} = d_{i\,i_2}$ then the same process can be repeated even getting $2^k d_{i_1i_1} = \dots = 2d_{i_ki_k} \in \{\alpha, \beta, d_{n-1n-1}\}$ for some $1 \le k \le n-3$. Therefore, one can conclude that $2^k(\alpha+\beta) \in \{\alpha, \beta,\}$ for some $1 \le k \le n-3$. This means that we can assume that $2^k(\alpha+\beta) = \alpha$. Result of this, we obtain

$d_{11} = \frac{\alpha}{2^{k+1}}, d_{i_1i_1} = \frac{\alpha}{2^k}, \dots, d_{i_ki_k} = \frac{\alpha}{2}, \beta = -\left(1 - \frac{1}{2^k}\right)\alpha$. Hence, $|\beta| < |\alpha|$ and obviously $2^s\beta \ne 2^r\alpha$ for any $r, s \in \mathbb{N}$.

Now, let $d_1 < \dots < d_p$ be the possible non-zero values of $|d_{22}|, \dots, |d_{n-2n-2}|$. Obviously, $\left\{\frac{|\alpha|}{2^k}, \dots, \frac{|\alpha|}{2}\right\} \subseteq \{d_1, \dots, d_{n-1}\}$. It is previously observed that that $\{2d_{22}, \dots, 2d_{n-2n-2}\} \subseteq spec(d)$ which means that $2d_1, \dots, 2d_p \in \{d_1, \dots, d_p, |\alpha|, |\beta|, |d_{n-1n-1}|\}$, as a result of this, $d_{ikik} = \frac{\alpha}{2}, 2d_p \le |\alpha|$ and $(2d_p \le |\beta|, 2d_p \le |d_{n-1n-1}|)$ we conclude that $d_p = \frac{|\alpha|}{2}, d_p = \frac{|\beta|}{2}$ or $d_p = \frac{|d_{n-1n-1}|}{2}$.

It should be known that there is one eigenvalues $d_{i_ki_k} = \frac{\alpha}{2}, d_{i_ki_k} = \frac{\beta}{2}$ or $d_{i_ki_k} = \frac{d_{n-2n-2}}{2}$ with module d_p. In reality, if there is i such that $|d_{ii}| = d_p, d_{ii} \ne \frac{\alpha}{2}$, then $spec(d) \ni 2d_{ii} \ne \alpha$ and $|2d_{ii}| = |\alpha|$. Therefore, there exists j such that $d_{jj} = 2d_{ii}$ afterwards $2d_{jj} \in spec(d)$ and $|2d_{jj}| = 2|\alpha| > |\alpha|$ which is a contradiction. Therefore, we have one eigenvalues with module d_p. By applying same process we deduce that there is one eigenvalues $\frac{1}{4}\alpha$ of module d_{p-1} and etc. Moreover, if all $d_2, \dots d_p$ are not in the form α also by the same process we can show that we have one eigenvalues with module d_p. Hence,

$\{d_{22}, \dots, d_{n-2n-2}\} \setminus \{0\} = \bigcup_{i=1}^{s}\left\{\frac{1}{2^i}\alpha\right\} \bigcup_{j=1}^{r}\left\{\frac{1}{2^j}d_{n-2n-2}\right\}$ or

$\{d_{22}, \dots, d_{n-2n-2}\} \setminus \{0\} = \bigcup_{i=1}^{s}\left\{\frac{1}{2^i}\alpha\right\} \bigcup_{j=1}^{r}\left\{\frac{d_{n-1n-1}}{2^j}\right\}$.

Case1.1.1: Let $\{d_{22}, \dots, d_{n-2n-2}\} \setminus \{0\} = \bigcup_{i=1}^{s}\left\{\frac{1}{2^i}\alpha\right\} \bigcup_{j=1}^{r}\left\{\frac{1}{2^j}d_{n-1n-1}\right\}$. We assume that $\{d_{s+1s+1}, \dots, d_{kk}\} \setminus \{0\} = \bigcup_{i=1}^{s}\left\{\frac{1}{2^i}\alpha\right\}$, $\{d_{k+1k+1}, \dots, d_{n-2n-2}\} \setminus \{0\} = \bigcup_{j=1}^{r}\left\{\frac{1}{2^j}d_{n-1n-1}\right\}$, and $d_{22} = \dots = d_{ss} = 0$ such that $|d_{s+1s+1}| \le \dots \le |d_{kk}|$, $|d_{k+1k+1}| \le \dots \le |d_{n-2n-2}|$. Firstly, one can realize that $d_{k+1k+1} \ne 2d_{ii}$ for all $1 \le i \le n$. Then, $a_{ik+1} = 0$ for all $1 \le i \le n$. Now, (9) and (7) yield $a_{i1} = \frac{\alpha - d_{11}}{d_{1n}} a_{in}$ which indicates that $1^{(st)}$ and $n^{(th)}$ columns of the matrix A are collinear and hence the other column should be non-zero and linearly independent. Due to assumption and (8) we deduce that $a_{ij} = 0$ if $2 \le i \le s$ and $s+1 \le j \le k, 2 \le i \le s$ and $k+1 \le j \le n-2, s+1 \le i \le k$ and $2 \le j \le s, s+1 \le i \le k$ and $k+1 \le j \le 2, k+1 \le i \le n-2$ and $2 \le j \le s, k+1 \le i \le n-2$ and $s+1 \le j \le k$. From $d_{s+1} \ne 2d_{ii}$ for all $2 \le i \le n-2$ and $rankA = n-2$ we obtain that $d_{s+1s+1} = 2d_{11}$.

Now, we are going to show that $\{d_{s+1s+1}, \dots, d_{kk}\}$ and $\{d_{k+1k+1}, \dots, d_{n-2n-2}\}$ do not contain an equal element. Suppose there exist an equal element, namely d_{s+2s+2}, d_{k+2k+2} to d_{s+1s+1}, d_{k+1k+1} respectively. Firstly, let us start with d_{s+2s+2} it is clear that $d_{s+2s+2} \ne d_{ii}$ for all $2 \le i \le n-2$. One can deduce that $a_{is+2} = 0$ for all $2 \le i \le n-2$ this is due to (8). Therefore, this means that the $(s+2)^{th}$ column is either zero or collinear to $(s+1)^{th}$ column of matrix A, which is a contradiction.

Now, suppose there exist an equal elements namely, d_{jj}, d_{j+1j+1} for some $s+2<j\le k-1$. Then, $2d_{ii}-d_{jj}=0$ if and only if $i=j-1$ and therefore (8) implies that $a_{ij}=0$ for all $i\ne j-1$ Again, in same manner $2d_{ii}-d_{j+1j+1}=0$ if and only if $i=j-1$ and therefore (8) yields that $a_{ij+1}=0$ for all $i\ne j-1$. This makes j^{th} and $(j+1)^{th}$ columns are collinear or at least one of them is zero, which contradicts to $rankA=n-2$.

By using the same process on $\{d_{k+1ks+1},\dots,d_{n-2n-2}\}$, one can show that all elements have different values. Hence, there are no an equal element among $\{d_{s+1s+1},\dots,d_{kk}\}$ and $\{d_{k+1k+1},\dots,d_{n-2n-2}\}$. Furthermore, $2^{(k-s)+1}d_{11}=2^{k-2}d_{s+1s+1}=\dots=2d_{kk}=\alpha$, $2^{(n-1)-(k+1)}d_{k+1k+1}=2^{(n-1)-(k+2)}d_{k+2k+2}=\dots=2d_{n-2n-2}=d_{n-1n-1}$ are simply derived. Which indicates that $d_{ii}=\dfrac{d_{n-1n-1}}{2^{(n-1)-i}}$ for all $k+1\le i\le n-2$ and $d_{11}=\dfrac{\delta}{2^{(k-s)+1}-1}$. Therefore, the matrix A should be in the following form:

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & a_{1s+1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{1n-1} & 0\\
0 & a_{22} & \cdots & a_{2s} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{2n-1} & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & a_{s} & \cdots & a_{ss} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{sn-1} & 0\\
0 & 0 & \cdots & 0 & 0 & a_{s+1s+2} & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{s+1n-1} & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{k-1k} & 0 & 0 & \cdots & 0 & a_{k-1n-1} & 0\\
a_{k1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{kn-1} & a_{kn}\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & a_{k+1k+2} & \cdots & 0 & a_{k+1n-1} & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{n-3n-2} & a_{n-3n-1} & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{n-2n-1} & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & \cdots & 0 & ba_{1s+1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}
\quad (A_1)
$$

Now, consider the following a sub matrix $\begin{pmatrix} a_{22} & \cdots & a_{2s}\\ \vdots & & \vdots\\ a_{s2} & \cdots & a_{ss}\end{pmatrix}\begin{pmatrix} d_{2n-1}\\ \vdots\\ d_{sn-1}\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$. Then, one can observe that all columns of the sub matrix A have to be linearly independent. This leads us to conclude that $d_{in-1}=0$ for all $2\le i\le s$. Hence,

- $d_{1n-1}=\dfrac{a_{kn-1}\left(2d_{kk}-d_{n-1n-1}\right)-a_{kn}d_{nn-1}}{a_{1n}}$
- $d_{1n-1}=\dfrac{a_{1n-1}}{a_{1j}}\left(2d_{ii}-d_{n-1n-1}\right)$, $s+1\le i\le n-2$
- $d_{i+1n-1}=\dfrac{a_{in-1}}{a_{ii+1}}\left(\dfrac{1}{2^{(n-1)-i-1}}-1\right)d_{n-1n-1}$, $k+1\le i\le n-2$

Case 1.1.2: Let $\{d_{22},\dots,d_{n-2n-2}\}\setminus\{0\}=\bigcup_{i=1}^{s}\left\{\frac{1}{2^i}\alpha\right\}\bigcup_{j=1}^{r}\left\{\frac{1}{2^j}\beta\right\}\quad\bigcup_{j_1=1}^{r_1}\left\{\frac{d_{n-1n-1}}{2^{j_1}}\right\}$.

Firstly, we assume that $\{d_{s+1s+1},\dots,d_{mm}\}\setminus\{0\}=\bigcup_{i=1}^{s}\left\{\frac{1}{2^i}\beta\right\}$, $\{d_{m+1m+1},\dots,d_{kk}\}\setminus\{0\}\bigcup_{i_1=1}^{s_1}\left\{\frac{1}{2^i}\alpha\right\}$, $\{d_{k+1k+1},\dots,d_{n-2n-2}\}\setminus\{0\}=\bigcup_{j=1}^{r}\left\{\frac{1}{2^j}d_{n-1n-1}\right\}$ and $d_{22}=\dots=d_{ss}=0$ such that $|d_{s+1s+1}|\le\dots\le|d_{mm}|$, $|d_{m+1m+1}|\le\dots\le|d_{kk}|$, $|d_{k+1k+1}|\le\dots\le|d_{n-2n-2}|$. Due to $2d_{ii}\ne d_{jj}$ such that $j\in\{s+1,k+1\}$, $a_{in}=ba_{i1}$ and (8) we derive that $a_{is+1}=a_{ik+1}=0$. Moreover, $rankA=n-2$ is satisfied i.e. the other columns are non zero and linearly independent. Similarly, as Case 1.1 we get that $d_{s+1s+1}\ne d_{s+2s+2}$ and so on. Therefore, $d_{mm}=2d_{m-1m-1}=\dots=2^{(m-(s+1))}d_{s+1s+1}$. This gives that $a_{ij}\ne 0$ for all $s+2\le j\le m$ such that $i\ne j-1$. Now, $2d_{ii}\ne d_{m+1m+1}$ for all $2\le j\le n-2$ and d_{m+1m+1} is defined by form $\dfrac{\alpha}{2^s1}$ and hence we conclude that $d_{m+1m+1}=d_{11}$. Otherwise $(m+1)^{th}$ column will be zero, which is a contradiction. By using the same way which is used in Case 1.1, we obtain that $d_{kk}=2d_{k-1k-1}=\dots=2^{(k-(m+1)+1)}d_{11}$. This implies $a_{ij}\ne 0$ for all $m+2\le j\le k$ such that $i\ne j-1$. In the same way, the following can be verified $d_{n-2n-2}=2d_{n-3n-3}=\dots=2^{(n-2-(k+1))}d_{k+1k+1}$. This

$a_{ij} \neq 0$ for all $k+2 \leq j \leq n-2$ such that $i \neq j-1$. Furthermore, if $s+1 \leq i \leq n-2$ and $2 \leq i \leq s$ then $a_{ij} = 0$ that is due to (8) and also $a_{nj} = ba_{1j}$ follows that $a_{nj} = 0$ for all $2 \leq i \leq s$. Hence, $d_{mm} = \frac{\beta}{2}, d_{kk} = \frac{\alpha}{2}$ and $d_{n-2n-2} = \frac{d_{n-1n-1}}{2}$. After simple calculations and dependence on (7), (8) we derive that

$$a_{mn} = \frac{d_{1n}}{\beta - d_{11}} a_{m1} \neq 0,\ a_{kn} = \frac{d_{1n}}{\alpha - d_{11}} a_{k1} \neq 0,$$

$d_{11} = \frac{\delta}{2^{k-(m+1)+2} - 1}$, $d_{s+1s+1} = \frac{1 - 2^{(k-(m+1)+1)}}{1 - 2^{m-(s+1)+1}} d_{11}$ and $d_{ii} = \frac{d_{n-1n-1}}{2^{(n-1)-i}}$, $k+1 \leq i \leq n-2$. Therefore, the matrix A should have the following form:

$$\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{1m+1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{1n-1} & 0 \\
0 & a_{22} & \cdots & a_{2s} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{2n-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & a_{s2} & \cdots & a_{ss} & 0 & 0 & \cdots & a_{m-1m} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{sn-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & a_{s+1s+2} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{s+1n-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{m-1m} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{m-1n-1} & 0 \\
a_{m1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{mn-1} & a_{mn} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & a_{m+1m+2} & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{m+1n-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{k-1k} & 0 & 0 & \cdots & 0 & a_{k-1n-1} & 0 \\
a_{k1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{kn=1} & a_{kn} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & a_{k+1k+2} & \cdots & 0 & a_{k+1n-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{n-3n-2} & a_{n-3n-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{n-2n-1} & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & ba_{1m+1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix} \quad (A_2)$$

Now, consider the following a sub matrix $\begin{pmatrix} a_{22} & \cdots & a_{2s} \\ \vdots & & \vdots \\ a_{s2} & \cdots & a_{ss} \end{pmatrix} \begin{pmatrix} d_{2n-1} \\ \vdots \\ d_{sn-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. Then, one can observe that all columns of the sub matrix A have to be linearly independent. This leads to conclude that $d_{in-1} = 0$ for all $2 \leq i \leq s$. Hence,

- $d_{m+1n-1} = \dfrac{a_{1n-1}(2d_{11} - d_{n-1n-1})}{a_{im+1}}$
- $d_{1n-1} = \dfrac{a_{in-1}(2d_{ii} - d_{n-1n-1}) a_{in} d_{nn-1}}{a_{i1}}$, $i \in \{m, k\}$
- $d_{i+1n-1} = \dfrac{a_{in-1}}{a_{ii+1}}(2d_{ii} - d_{n-1n-1})$, $s+1 \leq i \leq m-1$, $m+2 \leq i \leq k-1$
- $d_{i+1n-1} = \dfrac{a_{in-1}}{a_{ii+1}}\left(\dfrac{1}{2^{(n-1)-i-1}} - 1\right) d_{n-1n-1}$, $k+1 \leq i \leq n-2$

Case 1.2: Let $\alpha \neq 0, \beta = 0$. Evidently, one can check that $d_{11} = \delta$. We consider the possible non-zero values of $|d_{22}|, \ldots, |d_{n-2n-2}|$ such that $d_1 < \ldots < d_p$ and $d_{p+1} < \ldots < d_q$.

To keep away from duplicate steps, by tracking Case.1.1, one can deduce that $2d_1, \ldots, 2d_p, 2d_{p+1}, \ldots, 2d_q \in \{d_1, \ldots, d_p, |\alpha|, |d_{n-1n-1}|\}$ where $|\alpha| = 2d_p, d_p = 2d_{p-1}, \ldots d_2 = 2d_1$ and $|d_{n-1n-1}| = 2d_p, d_q = 2d_{q-1}, \ldots, d_{p+2} = 2d_{p+1}$. Moreover, as Case 1.1 one can see that there exists a unique eigenvalues $\frac{1}{\alpha}$ of module d_{p-1} and then for all eigenvalues. Thus,

$$spec(d) = \left\{\frac{1}{2p}\alpha, \ldots, \frac{1}{2}\alpha, \alpha, \frac{1}{2q} d_{n-1n-1}, \ldots, \frac{1}{2} d_{n-1n-1}\right\} \quad \text{or}$$

$$spec(d) = \left\{0, \frac{1}{2p}\alpha, \ldots, \frac{1}{2}\alpha, \alpha, \frac{1}{2q} d_{n-1n-1}\right\}.$$

By performing suitable basis permutation, we can assume that $|d_{22}| \leq \ldots \leq |d_{kk}|$, $|d_{k+1k+1}| \leq \ldots \leq |d_{n-2n-2}|$. Now, assume that we have $s-2$ and $r-2$ zeros among $\{d_{22}, \ldots, d_{kk}\}$, $\{d_{k+1k+1}, \ldots, d_{n-2n-2}\}$ respectively. Then, $\{d_{ii}\}_{n=2}^{s} \subseteq \{0\}$, $\{d_{ii}\}_{n=k+1}^{q} \subseteq \{0\}, d_{ss} < |d_{s+1s+1}| \leq |d_{s+2s+2}| \leq \ldots \leq |d_{kk}|$ and $d_{qq} < |d_{q+2q+2}| \leq \ldots \leq |d_{n-2n-2}|$ where $q = (r-2) - (k+1) + 1$. Now, due to (8), we obtain that $a_{ij} = 0$ if the followings are

satisfied:
$1\le i\le s$ and $s+1\le j\le k$, $2\le j\le s$ and $s+1\le i\le k$, $k+1\le i\le q$ and $q+1\le j\le n-2, q+1\le i\le n-2$ and $k+1\le j\le 1, 1\le i\le s$ and $q+1\le j\le k,\ k+1\le i\le q$ and $s+1\le j\le k$.

Clearly, $d_{jj}\ne 2d_{ii}$ for all $j\in\{s+1,q+1\}$, outcome for this and based on (8), we obtain $a_{is+1}=a_{iq+1}=0$ for all $1\le i\le n$ i.e., $(s+1)^{th}$ and $(q+1)^{th}$ columns of matrix A are zero.

Similarly, as Case 1.1 $\{d_{s+1s+1},\dots,d_{kk}\}$ and $\{d_{q+1q+1},\dots,d_{n-2n-2}\}$ can be shown that they do not contain an equal element. Hence, $d_{ii}=\frac{d_{n-1n-1}}{2^{(n-1)-i}}$ and $d_{11}=\delta$. In addition, (9) follows that $a_{i1}=a_{in}=0$ for all $s+1\le i\le k$, $q+1\le i\le n-2$. Hence, the matrix A should have the following form:

$$\begin{pmatrix}
a_{11} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{1n-1} & a_{1n}\\
a_{21} & a_{22} & \cdots & a_{2s} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{2n-1} & a_{2n}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
a_{s1} & a_{s2} & \dots & a_{ss} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{sn-1} & 0\\
0 & 0 & \cdots & 0 & 0 & a_{s+1s+2} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{s+1n-1} & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{k-1k} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{k-1n-1} & 0\\
a_{k1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{kn-1} & a_{kn}\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{k+1k+1} & \cdots & a_{k+11} & 0 & 0 & \cdots & 0 & a_{k+1n-1} & a_{k+1n}\\
\vdots & \vdots & & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{qk+1} & \cdots & a_{qq} & 0 & 0 & \cdots & 0 & a_{1n-1} & a_{qn}\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & a_{q+1q+2} & \cdots & 0 & a_{q+1n-1} & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{n-3n-2} & a_{n-3n-1} & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{n-2n-1} & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
ba_{1n} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix} \quad (A_2)$$

Now, consider the following a sub matrix

$$\begin{pmatrix} a_{22} & \cdots & a_{2s}\\ \vdots & & \vdots\\ a_{s2} & \cdots & a_{ss}\end{pmatrix}\begin{pmatrix} d_{2n-1}\\ \vdots\\ d_{sn-1}\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix},$$

$$\begin{pmatrix} a_{k+1k+1} & \cdots & a_{k+1q}\\ \vdots & & \vdots\\ a_{qk+1} & \cdots & a_{qq}\end{pmatrix}\begin{pmatrix} d_{k+1n-1}\\ \vdots\\ d_{qn-1}\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

One can observe that all columns of previous two sub matrix of A have to be linearly independent. This lead us to conclude that $d_{in-1}=0$ for all $2\le i\le s$ and $k+1\le i\le q$. Hence,

- $d_{1n-1}=\dfrac{a_{1n-1}\left(2d_{11}-d_{n-1n-1}\right)-a_{1n}d_{nn-1}}{a_{11}}$,

  $d_{kn-1}=\dfrac{a_{kn-1}\left(2d_{kk}-d_{n-1n-1}\right)-a_{kn}d_{nn-1}}{a_{k1}}$
- $d_{i+1n-1}=\dfrac{a_{in-1}}{a_{ii+1}}\left(2d_{ii}-d_{n-1n-1}\right)$, $s+1\le i\le k-1$,

  $q+1\le i\le n-2$
- $d_{i+1n-1}=\dfrac{a_{in-1}}{a_{ii+1}}\left(\dfrac{1}{2^{(n-1)-i+1}}-1\right)d_{n-1n-1}$, $q+1\le i\le n-3$.

Case 1.3: Let $\alpha\beta\ne 0, \alpha=-\beta$ i.e., $d_{11}=0$. By applying the same approach which used in Case 1.1 of this Lemma we obtain that $2d_1,\dots,2d_p\in\{d_1,\dots,d_p,|\alpha|,|d_{n-1n-1}|\}$, there exists eigenvalues $\frac{\alpha}{2},-\frac{\alpha}{2}$ or d_{n-1n-1} with module d_p and so for all eigenvalues. Therefore, we can assume that $\{d_{22},\dots,d_{kk}\}\setminus\{0\}=\bigcup_{j_1=1}^{r_1}\left\{\frac{1}{2^{j_1}}\alpha\right\}\bigcup_{j_2=1}^{r_2}\left\{\frac{1}{2^{j_2}}\alpha\right\}$,

$\{d_{k+1k+1},\dots,d_{n-2n-2}\}\setminus\{0\}=\bigcup_{i=1}^{s}\left\{\frac{1}{2^i}d_{n-1n-1}\right\}$ such that $|d_{22}|\le\ldots\le|d_{kk}|$, $|d_{k+1k+1}|\le\ldots\le|d_{n-2n-2}|$. First of all, one can see that $d_{k+1k+1}\ne 2d_{jj}$ for all $1\le i\le n$ and this means that $a_{k+1j}=0$ for all $1\le i\le n$. Now, let $\frac{1}{2}\alpha,-\frac{1}{2}\alpha\notin\{d_{22},\dots,d_{kk}\}$. Due to (9), we obtain $1^{(st)}$ and $n^{(th)}$ columns are zero, which contradicts $rankA=n-2$. Hence, there has to exist $2\le m\le k$ such that $d_{mm}\in\left\{\frac{1}{2}\alpha,-\frac{1}{2}\alpha\right\}$.

Now, let $\{d_{22},\ldots,d_{k+1k+1}\}\setminus\{0\}\supseteq\bigcup_{j_2=1}^{r_2}\left\{\frac{1}{2^{j_2}}\alpha\right\}$, $-\frac{1}{2}\alpha$ $\notin\{d_{22},\ldots,d_{k+1k+1}\}$. Then, from (9) and (7), one can see that $a_{i1}=\frac{\alpha}{d_{1n}}a_{in}$ i.e., $rankA=n-2$ is satisfied. Depending on assumption and (8), there exists $d_{pp}=\frac{1}{2^s}\alpha$, this makes $p^{(th)}$ column is zero which is a contradiction. In the same way, we obtain a contradiction to $rankA=n-2$ if we suppose that $\{d_{22},\ldots,d_{k+1k+1}\}\setminus\{0\}\supseteq\bigcup_{j_2=1}^{r_2}\left\{-\frac{1}{2^{j_2}}\alpha\right\}$ and $-\frac{1}{2}\alpha\notin\{d_{22},\ldots,$ $d_{k+1k+1}\}$. Now, if

$$\{d_{22},\ldots,d_{k+1k+1}\}\setminus\{0\}=\bigcup_{j_1=1}^{r_1}\left\{-\frac{1}{2^{j_1}}\alpha\right\}\bigcup_{j_2=1}^{r_2}\left\{\frac{1}{2^{j_2}}\alpha\right\}$$

Clearly, there p and q such that $d_{pp}=\frac{1}{2^{r_2}}\alpha$, $d_{qq}=\frac{1}{2^{r_1}}\alpha$. Also, (8) yields that $p^{(th)}$ and $q^{(th)}$ columns have to equal to zero which is a contradiction to $rankA=n-2$. Hence, this case is impossible.

Case2: Let $d_{1n}\neq 0$. It is easy to find $spec(d)=\{d_{11},\ldots,d_{n-1n-1}\}$. Now, one can see that $spec(d)=\{d_{11},\ldots,d_{n-2n-2}\}\supseteq\{2d_{11},2d_{22}\ldots,2d_{n-2n-2}\}$. Let $\lambda\in spec(d)$ be such that $|\lambda|=\max_{1\le i\le n}|d_{ii}|$. Clearly, if $\lambda\in\{d_{11},\ldots,d_{n-2n-2}\}$ then $2\lambda\in spec(d)$, this gives that $\lambda=0$. Therefore, $d_{11}=\ldots=d_{n-1n-1}=0$ and $d(e_i)=d_{in-1}e_{n-1}$ for all $1\le i\le n-2$. Then, (2) follows that $\sum_{j=1}^{n}a_{ij}d_{jn-1}e_{n-1}=d(e_i^2)=0$ for all $1\le i\le n-2$. The last one implies that vector $(d_{1n-1},\ldots,d_{n-2n-1},0,d_{nn-1})$ is a solution of homogeneous linear system of equations $Ax=0$. We note that $a_{in}=ba_{i1}$ and therefore if the first $n-2$ columns are linearly independent, then $d=0$. In order to $d\neq 0$ we consider the matrices (A_4) which has the first $n-2$ columns linearly dependent. Thus, a result of this d is in the form (D_4).

Now, let $\lambda\notin\{d_{11},\ldots,d_{n-2n-2}\}$ then $\lambda=d_{n-1n-1}$ and we can assume that $d_{n-1n-1}\neq 0$. Let $d_1<\ldots<d_p$ be the possible non-zero values of $|d_{11}|,\ldots,|d_{n-2n-2}|$. According to $\{2d_{11},\ldots,2d_{n-2n-2}\}\subseteq spec(d)$ one can see that $2d_1,\ldots,2d_p\in\{d_1,\ldots,d_p,|d_{n-1n-1}|\}$. Moreover, as these values are non-zero, we conclude that $|d_{n-1n-1}|=2d_p, d_p=2d_{p-1},\ldots,d_2=2d_1$. Similarly, as Case 1.1.1 one can show that there can be only one eigenvalue $\frac{1}{2}d_{n-1n-1}$ with module d_p. As well one obtains that there is only one eigenvalues $\frac{1}{4}d_{n-1n-1}$ of module d_{p-1} and etc.

Hence, $spec(d)=\left\{\frac{d_{n-1n-1}}{2p},\ldots,\frac{d_{n-1n-1}}{2},d_{n-1n-1}\right\}$ or $spec(d)=\left\{0,\frac{d_{nn}}{2p},\ldots,\frac{d_{n-1n-1}}{2},d_{n-1n-1}\right\}$. Now, by performing appropriate basis permutation $|d_{11}|\le\ldots\le|d_{n-2n-2}|<|d_{n-1n-1}|$ can be assumed. Now, (2) $\sum_{j=1}^{n}a_{ij}d_{jn-1}=2d_{ii}a_{in=1}$ and $a_{ij}(2d_{ii}-d_{jj})$, for all $1\le i,j\le n-2$.

Assume that there exist k zeros among $d_{11},\ldots,d_{n-2n-2}$. Then $0=d_{11}=\ldots=d_{kk}<|d_{k+1k+1}|\le\ldots\le|d_{n-2n-2}|<|d_{n-1n-1}|$. Furthermore, $a_{ij}(2d_{ii}-d_{jj})=0$ follows that $a_{ij}=0$ if $1\le i\le k$ and $k+1\le j\le n-2, 1\le j\le k$ and $k+1\le i\le n-2$.

One can see that $d_{k+1k+1}\neq 2d_{ii}$ for all $k+1\le i\le n-2$, based on (10) we obtain $a_{ik+1}=0$ for all $k+1\le i\le n-2$ i.e., the $(k+1)^{th}$ column of matrix A is zero. In order to get $rankA=n-2$ we can consider the $(k)^{th}$ column of matrix A is also equal to zero. Therefore, the other columns should be non-zero and linearly independent. By implementing same process which is applied in Case 1.1.1 one can check that $d_{k+1k+1},\ldots,d_{n-1n-1}$ has no an equal element. Hence in this case all $d_{k+1k+1},\ldots,d_{n-1n-1}$ are distinct and $d_{ii}=\frac{d_{n-1n-1}}{2^{(n-1)-i}}$ for all $k+1\le i\le n-2$. Now if $k+1\le i,j\le n-2$ then (10) is equal to zero if and only if $j=i+1$ and hence $a_{ij}=0$ for all $k+1\le i,j\le n-1$. Therefore, the matrix A should have the following form:

$$\begin{pmatrix} a_{11} & \cdots & a_{1k-1} & 0 & 0 & 0 & \cdots & 0 & a_{1n-1} & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & \cdots & a_{kk-1} & 0 & 0 & 0 & \cdots & 0 & a_{kn-1} & a_{kn} \\ 0 & \cdots & 0 & 0 & 0 & a_{k+1k+2} & \cdots & 0 & a_{k+1n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & a_{n-3n-2} & a_{n-3n-1} & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & a_{n-2n-1} & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \quad (A_5)$$

Now, consider the following a sub matrix $\begin{pmatrix} a_{11} & \cdots & a_{2k-1} & 0 \\ \vdots & & \vdots & \vdots \\ a_{k2} & \cdots & a_{kk-1} & 0 \end{pmatrix}\begin{pmatrix} d_{1n-1} \\ \vdots \\ d_{kn-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. Then, one can observe that all columns of the sub matrix A have to be linearly independent. This leads us to conclude that $d_{in-1} = 0$ for all $1 \le i \le k$. Hence,

$$d_{i+1n-1} = \frac{a_{in-1}}{a_{ii+1}}\left(\frac{1}{2^{(n-1)i-1}} - 1\right) d_{n-1n-1},\ a_{ii+1} \ne 0,\ k+1 \le i \le n-2$$

.

Corollary: *Let evolution algebra has a matrix* $A = (a_{ij})_{1 \le i,j \le n}$ *in the natural basis* $e_1, \dots, e_n$ *such that* $e_n^2 = 0, \quad e_{n-1}^2 = \sum_{k=1}^{n-1} b_k e_k^2,\ b_p \ne 0$ *for some* $1 \le p \ne q \le n$, $rankA = n-2$. *Then derivation* d *of this evolution algebra is either zero or it is in one of the following forms up to basis permutation:*

$$\begin{pmatrix} 0 & \cdots & 0 & d_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & d_{n-1n} \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (D_{10})$$

Where $\sum_{k=1}^{n-2} a_{ik} d_{kn} = 0,\ 1 \le i \le n-1;$

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \frac{d_{nn}}{2^{n-k-1}} & \cdots & 0 & d_{k+1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{d_{nn}}{2} & d_{n-1n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & d_{nn} \end{pmatrix}, \quad (D_{11})$$

Where

$$d_{i+1n} = \frac{a_{in}}{a_{ii+1}}\left(\frac{1}{2^{n-i-1}} - 1\right) d_{nn},\ a_{ii+1} \ne 0,\ k+1 \le i \le n-2,$$

$1 \le k \le n-1$ *and* $d_{k+1n} \in$ □ . *The matrix* A *should be in the following form:*

$$\begin{pmatrix} a_{11} & \cdots & a_{1k-1} & 0 & 0 & 0 & \cdots & 0 & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{k1} & \cdots & a_{kk-1} & 0 & 0 & 0 & \cdots & 0 & a_{kn} \\ 0 & \cdots & 0 & 0 & 0 & a_{k+1k+2} & \cdots & 0 & a_{k+1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & a_{n-2n-1} & a_{n-2n} \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & a_{n-1n} \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (A_6)$$

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