International Journal of Engineering Trends and Technology
ISSN: 2231 – 5381 / https://doi.org/10.14445/22315381/IJETT-V70I9P205

Volume 70 Issue 9, 47-56, September 2022

Original Article

Computational Approach via Half-Sweep Similarity and SOR Schemes for 2D Hyperbolic Telegraph Equations

Nur Afza Mat Ali[1], Jumat Sulaiman[2], Azali Saudi[3], Nor Syahida Mohamad[4]

[1,2,4]*Faculty of Science and Natural Resources, Universiti Malaysia Sabah, Kota Kinabalu, Malaysia*
[2]*Preparatory Centre for Science and Technology, Universiti Malaysia Sabah, Kota Kinabalu, Malaysia*
[3]*Faculty of Computing and Informatics, Universiti Malaysia Sabah, Kota Kinabalu, Malaysia*

[1]*Corresponding Author : afzamatali@yahoo.com*

Received: 20 June 2022 Revised: 31 July 2022 Accepted: 13 August 2022 Published: 06 September 2022

Abstract *- This paper uses the similarity transformation to get similar solutions to the two-dimensional hyperbolic telegraph equation (2-D HTE). The similarity solution of 2-D HTE is then derived using a rotated five-point similarity finite difference (SFD) discretization scheme to obtain the rotated five-point SFD approximation equation. Gathering rotated approximation equations generate a linear system with large-scale and sparse matrix characteristics. Since then, the linear system has been solved using the half-sweep similarity technique via half-sweep successive over-relaxation (HSSOR-SFD) iteration. Three numerical examples are presented in this paper to validate the performance of the HSSOR-SFD iteration in solving 2-D HTE. The numerical findings showed that the version of HSSOR-SFD iteration is the best compared to the standard similarity methods: full-sweep Gauss-Seidel (FSGS-SFD) and full-sweep SOR (FSSOR-SFD) iterations in terms of iteration number and computational time.*

Keywords *- Two-dimensional hyperbolic telegraph equation, Similarity transformation, Similarity solution, Rotated similarity finite difference method, Similarity half-sweep SOR method.*

1. Introduction

Hyperbolic partial differential equations (HPDEs) play a crucial role as a mathematical model in the various fields of science and engineering. These equations are interesting as they are modeled to understand various physical and complex phenomena. For instance, the mechanical wave [1], cosmic-ray transport [2], random walk theory [3], communication system [4] as well as signal analysis [5] are modeled by HPDE. In this study, we fix attention on finding a numerical solution to the two-dimensional hyperbolic telegraph equation (2-D HTE) as follows:

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha\frac{\partial u}{\partial t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + F(x,y,t) \qquad (1)$$

subject to the initial conditions

$$u(x,y,0) = u_0(x,y);\ \frac{\partial u}{\partial t}(x,y,0) = v_0(x,y);\ (x,y) \in \Omega \qquad \ldots(2)$$

and boundary conditions

$$B[u(x,y,t)] = g(x,y,t), (x,y,t) \in \partial\Omega \times [0,T] \qquad (3)$$

where α and β are given constants, $F(x,y,t)$ is given function. This kind of equation has proved useful worldwide because of its rife applications. However, it is quite difficult to find the analytical solution to the problem (1) and also suffers from high computational costs due to the high number of variables involved. Therefore, various studies have been found to reduce the number of variables in an equation so that simpler to solve numerically and results in less computational complexity.

Among the methods that have been used for solving HPDE problems in the context of reducing the variables from an original equation are similarity methods such as the differential quadrature method (DQM) [6-9], Chebyshev spectral collocation method [10], and method of lines [11]. In articles [6], Singh et al. applied the hybrid cubic B-spline differential quadrature method to reduce the one-dimensional telegraph equation into a system of 1st order ordinary differential equations (ODE). Next, Mittal and Bhatia [8] used a modified cubic B-spline differential quadrature method to convert a 2-D HTE into a system of first-order ODE. Then, the author of [10] used Chebyshev differentiation matrices to transform a system of PDEs with initial conditions into a system of ODEs for telegraph equations. Afterwards, the solutions were obtained in [11] using the lines method adopted by Jator in reducing hyperbolic and elliptic PDEs into systems of boundary value problems and initial value problems in ODEs. In general, the purpose of the similarity technique is to convert a given problem into a simpler problem, either transform the PDE into a system of ordinary differential equations (ODEs) or reduce at least one variable from the original equation and therefore make the solution much simpler. It was found that applying the similarity technique to PDE problems made the results simpler and low computational cost while maintaining good accuracy. Due to the advantages, the similarity methods are used to solve various problems, see [12-16]. Since then, the approach of similarity methods has inspired us to apply it to get similar solutions for the two-dimensional HTE problem. However, these similar solutions for PDE's problem still require numerical solutions since the analytical solutions are quite hard to get. Therefore, various numerical solutions are actively being found by suggesting new numerical approximation techniques to solve the HPDE problems.

The most classical techniques for solving HPDE problems are finite difference, finite element, and finite volume. In article [17], the generalized finite difference method is used to solve the telegraph equation numerically in 2-D and 3-D spaces. Next, Aslefallah and Rostamy [18] applied the singular boundary method to obtain a numerical solution for the 2-D telegraph equation on an arbitrary domain. Based on the study in [19], a new spectral Galerkin method is employed to solve second-order HTE in 2-D spaces. Then, Devi et al. [20] used Lagrange's operational approach to find the numerical solution of 2-D HTE subject to Dirichlet boundary conditions. In [21], they constructed a hybrid meshless method for the solution of the two-dimensional HTE with Dirichlet or mixed boundary conditions. In [22], the extension of the generalized finite difference to the explicit solution of hyperbolic and parabolic equations was developed for PDE with constant coefficients when considering 1-,2- or 3-D spaces. After that, Ding and Zhang [23] applied a compact finite difference scheme for solving 2-D second-order non-homogeneous linear hyperbolic equations. Combining the original finite difference scheme with other approaches inspired us to introduce a new finite difference scheme, namely the rotated five-point similarity finite difference (SFD) scheme, which comes from a combination of similarity techniques via a rotated five-point finite difference scheme. A rotated five-point SFD scheme is used to discretize the differential part in the similarity solution of the 2-D HTE problem to get a rotated five-point SFD approximation equation and then generate a linear system with the characteristic large-scale and sparse.

According to previous research about a large-scale and sparse linear system solution, the most effective solver is the iterative method [24-26]. In recent years, many researchers have applied some iterative methods such as GS [27, 28], SOR [29-32], KSOR [33, 34], AOR [35], and AGE [36, 37], to obtain the numerical solution of the linear system. However, the computational complexity is still high since it needs more computational time to satisfy the convergence criteria. Due to its weaknesses, Abdullah [38] proposed the concept of HS iteration in 1991 to solve the 2-D Poisson equation. The author has shown that the main idea behind the HS iteration concept is to take only half of the integer points in the solution area of the proposed problem. Due to the advantages, the HS iteration is used to solve various problems, see [39-43]. Therefore, the concept of HS iteration can potentially reduce the computational complexity in the solution procedure, which naturally leads to several iterations and faster computational time. Motivated by the unique property of HS in securing a low computational complexity while computing the numerical solutions, the further discussion of this article will focus on the use of the HS concept combined with the similarity finite difference method via successive over-relaxation (SOR) method, namely HSSOR-SFD method to realize the concept of HS iteration while obtaining the numerical solution of the linear system, which is generated from rotated five-point SFD approximation equation.

So, the main motivation of the present study is to develop numerical solutions for the 2-D HTE using a similarity technique combined with a finite difference scheme, namely a rotated five-point SFD scheme via HSSOR iterative method. For this purpose, similarity transformation is used to reduce 2-D HTE into a 2-D elliptic telegraph equation (2-D ETE), which becomes simpler to solve numerically than the original equation since the independent variables of 2-D HTE are reduced. The new five-point SFD scheme is used to discretize the differential terms of 2-D ETE to get the rotated five-point SFD approximation solutions. A large-scale and sparse linear system constructed from a rotated five-point SFD approximate equation is then solved iteratively using the HSSOR-SFD method.

2. Similarity Solutions

In this section, the number of independent variables in eq. (1) is reduced by applying similarity transformation in particular wave variables transformation to get the similarity solutions of 2-D HTE. First, the wave variables are introduced as follows [44, 45]

$$u(x, y, t) = U(\xi, \tau);\ \xi = x - ct;\ \tau = y - dt \qquad (4)$$

Where c and d are constant. Then, by putting eq. (4) into eq. (1) and after some simple calculating, the following similar solution of 2-D HTE is reached, namely two-dimensional elliptic telegraph equation (2-D ETE), which can be written as:

$$v_1\left(\frac{d^2u}{d\xi^2} + \frac{d^2u}{d\tau^2}\right) - v_2\left(\frac{du}{d\xi} + \frac{du}{d\tau}\right) + \beta^2 u(\xi, \tau) = f(\xi, \tau) \qquad \ldots(5)$$

with $v_1 = \frac{\eta^2}{2} - 1$, $v_2 = 2\alpha\frac{\eta}{2}$ and $c = d = \eta$. Since eq. (5) still has the differential part; the discretization is needed using a new finite difference scheme known as the rotated five-point similarity finite difference (SFD) discretization scheme to get the rotated five-point SFD approximation equation, which discusses in the next section.

3. Rotated Five-Point SFD Approximation Equation

As mentioned in the section before, the differential part of the similarity solution of 2-D HTE in eq. (5) is discretized by using new discretization scheme formula, namely a rotated five-point similarity finite difference (SFD) discretization scheme to get the rotated five-point SFD approximation equation. The rotated five-point SFD approximation equation is based on the cross-orientation operator, which can be obtained by rotating the ξ-plane axis and τ-plane axis clockwise 45° for the original mesh, see Figure 3.

So, to start the discussion of approximating the similarity solution of 2-D HTE in eq. (5), a new scheme formula is introduced, which is a rotated five-point SFD scheme as follows:

$$\left.\frac{du}{d\xi}\right|_{\vartheta,\varphi} = \frac{U_{\vartheta+1,\varphi+1}-U_{\vartheta-1,\varphi+1}+U_{\vartheta+1,\varphi-1}-U_{\vartheta-1,\varphi-1}}{4h}$$
$$\left.\frac{du}{d\tau}\right|_{\vartheta,\varphi} = \frac{U_{\vartheta+1,\varphi+1}-U_{\vartheta+1,\varphi-1}+U_{\vartheta-1,\varphi+1}-U_{\vartheta-1,\varphi-1}}{4h}$$
$$\left.\frac{d^2u}{d\xi^2}+\frac{d^2u}{d\tau^2}\right|_{\vartheta,\varphi} = \frac{U_{\vartheta+1,\varphi+1}+U_{\vartheta+1,\varphi-1}+U_{\vartheta-1,\varphi+1}+U_{\vartheta-1,\varphi-1}-4U_{\vartheta,\varphi}}{2h^2}$$
…(6)

Then, by substituting eq. (6) into eq. (5) this will result in the rotated five-point SFD approximation equation as follows:

$$v_1\left(\frac{U_{\vartheta+1,\varphi+1}+U_{\vartheta+1,\varphi-1}+U_{\vartheta-1,\varphi+1}+U_{\vartheta-1,\varphi-1}-4U_{\vartheta,\varphi}}{2h^2}\right) - v_2\left(\frac{U_{\vartheta+1,\varphi+1}-U_{\vartheta-1,\varphi+1}+U_{\vartheta+1,\varphi-1}-U_{\vartheta-1,\varphi-1}}{4h} + \frac{U_{\vartheta+1,\varphi+1}-U_{\vartheta+1,\varphi-1}+U_{\vartheta-1,\varphi+1}-U_{\vartheta-1,\varphi-1}}{4h}\right) + \beta^2 U_{\vartheta,\varphi} = f_{\vartheta,\varphi}$$
…(7)

By simplifying and reordering eq. (7), we obtain

$$\frac{\phi_1-2\phi_2}{\beta^2-4\phi_1}U_{\vartheta+1,\varphi+1} + \frac{\phi_1}{\beta^2-4\phi_1}U_{\vartheta+1,\varphi-1} + \frac{\phi_1}{\beta^2-4\phi_1}U_{\vartheta-1,\varphi+1} + \frac{\phi_1+2\phi_2}{\beta^2-4\phi_1}U_{\vartheta-1,\varphi-1} + U_{\vartheta,\varphi} = \frac{f_{\vartheta,\varphi}}{\beta^2-4\phi_1}$$
…(8)

where $\phi_1 = \frac{v_1}{2h^2}$ and $\phi_2 = \frac{v_2}{4h}$.

A rotated five-point SFD scheme is built by splitting the solution domain into two points categories on the ξ, τ-plane. The evaluation of eq. (8) may depend on the only type (○) of points. Iterations involving only one type of point can then be generated. Once the convergence test is reached, the solution at the remaining points ( ) will be assessed directly once using the original centred five-point SFD approximation equation as follows:

$$\frac{\phi_{1*}+\phi_{2*}}{\beta^2-4\phi_{1*}}U_{\vartheta-1,\varphi} + \frac{\phi_{1*}-\phi_{2*}}{\beta^2-4\phi_{1*}}U_{\vartheta+1,\varphi} + \frac{\phi_{1*}+\phi_{2*}}{\beta^2-4\phi_{1*}}U_{\vartheta,\varphi-1} + \frac{\phi_{1*}-\phi_{2*}}{\beta^2-4\phi_{1*}}U_{\vartheta,\varphi+1} + U_{\vartheta,\varphi} = \frac{f_{\vartheta,\varphi}}{\beta^2-4\phi_{1*}}$$
…(9)

where $\phi_{1*} = \frac{v_1}{h^2}$ and $\phi_{2*} = \frac{v_2}{2h}$.

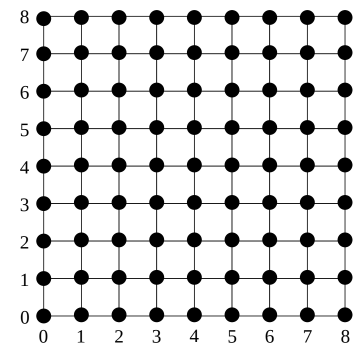


Fig. 1 Finite grid networks for the full-sweep in case $m = 8$

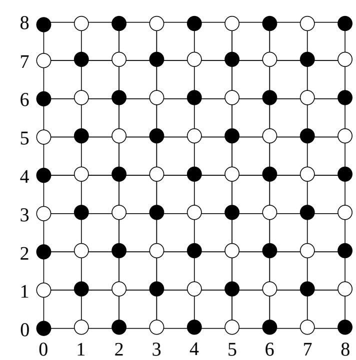


Fig. 2 Finite grid networks for the half-sweep in case $m = 8$

As to the computational molecule of both full- and half-sweep techniques, Fig. 3 and 4 show the information where the full-sweep technique has a standard computational molecule for finite difference approximation while the half-sweep technique has the cross-orientation computational molecule as opposed to the standard computational molecule.

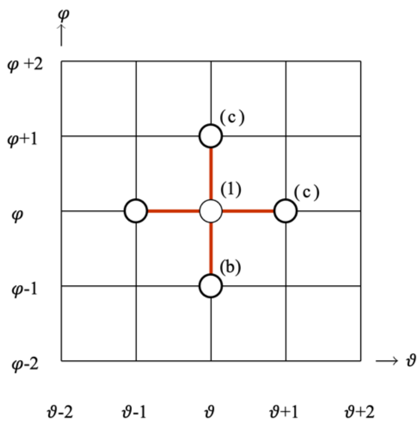


Fig. 3 The computational molecules for centred five-point SFD approximation

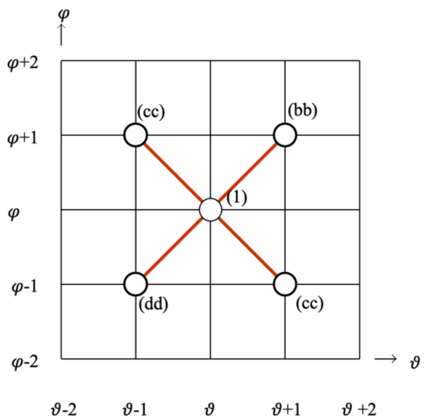


Fig. 4 The computational molecules for rotated five-point SFD approximation

where

$b = \frac{\phi_{1*}+\phi_{2*}}{\beta^2-4\phi_{1*}}$

$c = \frac{\phi_{1*}-\phi_{2*}}{\beta^2-4\phi_{1*}}$

$bb = \frac{\phi_1-2\phi_2}{\beta^2-4\phi_1}$

$cc = \frac{\phi_1}{\beta^2-4\phi_1}$

$dd = \frac{\phi_1+2\phi_2}{\beta^2-4\phi_1}$

Now considering a certain number of grid points based on eq. (8) as depicted in Fig. 2, the construction of the large-scale and sparse linear system is obtained, which can be expressed in the matrix form as follows:

$$W\underline{U} = \underline{S} \tag{10}$$

where W is a square nonsingular matrix while U and S are column matrices. We will use iterative methods in the next section to solve the linear system (10).

4. Formulation of Proposed Iterative Method

In this section, the half-sweep similarity technique via half-sweep successive over-relaxation (HSSOR-SFD), full-sweep Gauss-Seidel (FSGS-SFD) and full-sweep SOR (FSSOR-SFD) iterations are formulated to solve the large-scale and sparse linear system in eq. (10) as noted in the previous section. Firstly, start with how to derive the formulation of the GS iterative method. To derive the formulation of the proposed iterative method, the coefficient matrix W of the linear system (10) decomposes as:

$$W = D - R - V \tag{11}$$

where D, R and V represent diagonal, lower triangular, and upper triangular matrices, respectively. The formulation of GS iteration can be stated in vector form (12) by applying the decomposition in eq. (11) into the linear system (10) as follows [24, 25, 30, 31]:

$$\underline{U}^{(k+1)} = (D-R)^{-1}V\underline{U}^{(k)} + (D-R)^{-1}\underline{S} \tag{12}$$

For the implementation of GS iteration, each component ${U_{\vartheta,\varphi}}^{(k+1)}$ can be computed as:

$${U_{\vartheta,\varphi}}^{(k+1)} = S_{\vartheta,\varphi} - a\left({U_{\vartheta-1,\varphi}}^{(k+1)} + U_{\vartheta,\varphi-1}\right) - b\left({U_{\vartheta+1,\varphi}}^{(k+1)} + U_{\vartheta,\varphi+1}\right) \quad \ldots(13)$$

Next, implementing a parameter ω is considered a relaxation factor to derive the formulation of the SOR iteration. By substituting ω into eq. (10) and rewrite it as [46]:

$$\omega W\underline{U} = \omega\underline{S} \tag{14}$$

The optimal value of ω in the range [1,2). Then, by applying the decomposition in eq. (11) into the linear system (14), the iterative formulation of the SOR method can be stated in vector form as [47, 48]:

$$\underline{U}^{(k+1)} = (D-\omega R)^{-1}(\omega V + (1-\omega)D)\underline{U}^{(k)} + \omega(D-\omega R)^{-1}\underline{S} \quad \ldots(15)$$

Note that for $\omega = 1$, eq. (15) is turning out to be the GS method. To be specific, a good choice of ω can dramatically improve the computational performance of the SOR iteration. For the implementation of the SOR iteration, each component ${U_{\vartheta,\varphi}}^{(k+1)}$ can be computed as:

$${U_{\vartheta,\varphi}}^{(k+1)} = (1-\omega){U_{\vartheta,\varphi}}^{(k)} + \omega\left(\frac{f_{\vartheta,\varphi}}{\beta^2-4\phi_{1*}} - \frac{\phi_{1*}+\phi_{2*}}{\beta^2-4\phi_{1*}}\left({U_{\vartheta-1,\varphi}}^{(k+1)} + {U_{\vartheta,\varphi-1}}^{(k+1)}\right) - \frac{\phi_{1*}-\phi_{2*}}{\beta^2-4\phi_{1*}}\left({U_{\vartheta+1,\varphi}}^{(k+1)} + {U_{\vartheta,\varphi+1}}^{(k+1)}\right)\right) \quad \ldots(16)$$

To improve the computational performance, half-sweep iterative methods based on the rotated finite difference approximations are much faster than the methods based on the standard five-point formula in solving the partial differential equations due to the formers' overall lower computational complexities [39, 49, 50]. So, in the same way, the implementation of HSSOR iteration for each component ${U_{\vartheta,\varphi}}^{(k+1)}$ can be computed as:

$$ {U_{\vartheta,\varphi}}^{(k+1)} = (1-\omega){U_{\vartheta,\varphi}}^{(k)} + \omega * \left(\frac{f_{\vartheta,\varphi}}{\beta^2-4\phi_1} - \frac{\phi_1-2\phi_2}{\beta^2-4\phi_1}{U_{\vartheta+1,\varphi+1}}^{(k+1)} - \frac{\phi_1}{\beta^2-4\phi_1}\left({U_{\vartheta+1,\varphi-1}}^{(k+1)} + {U_{\vartheta-1,\varphi+1}}^{(k+1)}\right) - \frac{\phi_1+2\phi_2}{\beta^2-4\phi_1}{U_{\vartheta-1,\varphi-1}}^{(k+1)}\right) \quad \ldots(17) $$

Using the half-sweep technique, the computational complexity was reduced to half due to only half of the node points being taken into account when performing calculations. Fig. 1 shows the computational grid for both full-sweep and half-sweep techniques to understand the half-sweep concept better.

5. Numerical Experiments and Discussions

In this section, three problems of the two-dimensional hyperbolic telegraph equation (2-D HTE) are used to show the performance of the HSSOR-SFD iteration. The experiments were executed on a Macbook Air with an Apple M1 chip. We considered three parameters which are the number of iterations (*iter*), computational time (*t*), and maximum absolute error (*err*), in making a comparison of the proposed iterative methods. The rate of tolerance error for the convergence test is set to $\varepsilon = 10^{-10}$ To check the consistency, we tend to run the numerical simulation by increasing the values of grid sizes $64, 128, 256, 512$ and 1024. Below are the following three 2-D HTE problems:

Prob. 1 [51] Consider the 2D-HTE in eq. (1) with $\alpha = \beta = 1$ in the domain $[0,1] \times [0,1]$, the analytical solution is $u(x,y,t) = e^{x+y-t}$ and the source function is $F(x,y,t) = -2e^{x+y-t}$.

Prob. 2 [52] Consider the 2D-HTE in eq. (1) with $\alpha = 2, \beta = 1$ in the domain $[0,2] \times [0,2]$, the analytical solution is $u(x,y,t) = e^{-t}\sin(x)\,y^2$ and the source function is $F(x,y,t) = -e^{-t}\sin(x)\,(y^2+2)$.

Prob. 3 [53] Consider the 2D-HTE in eq. (1) with $\alpha = 10, \beta = 5$ in the domain $[0,1] \times [0,1]$, the analytical solution is $u(x,y,t) = exp(-t)\sinh x \sinh y$ and the source function is $F(x,y,t) = (-2\alpha + \beta^2 - 1)exp(-t)\sinh x \sinh y$.

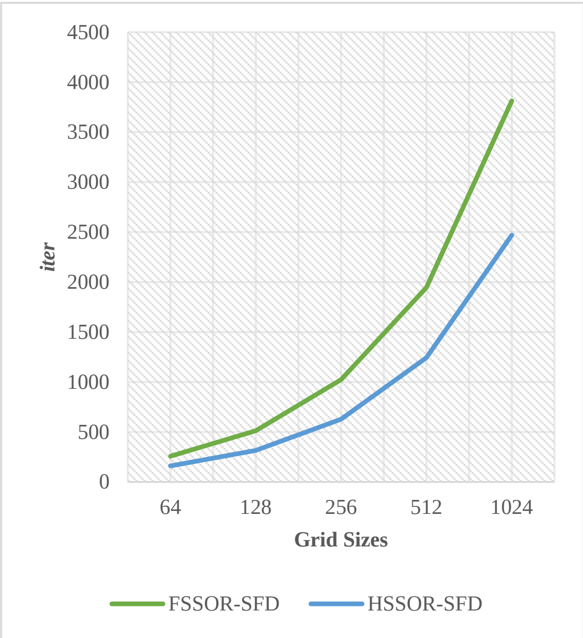
All significant numerical results for numerical experiments of solving examples 1, 2, and 3 using the HSSOR-SFD, FSSOR-SFD and FSGS-SFD iterations are recorded in Tables 1, 2, and 3, respectively. Based on the observation in Tables 1, 2 and 3, it is found that the HSSOR-SFD iterative method requires a smaller number of iterations as compared with FSSOR-SFD and FSGS-SFD. In terms of execution time, the HSSOR-SFD iterative method provides significantly faster performance than FSSOR-SFD and FSGS-SFD. The proposed HSSOR-SFD iterative method is more efficient than FSSOR-SFD and FSGS-SFD iterative methods.

The following is a summary of the findings. For problem 1, the *iter* and *t* of the HSSOR-SFD iteration have declined by about 96.69% - 99.67% and 95.03% - 99.56%, respectively, compared to the FSGS-SFD iteration. Meanwhile, if FSSOR-SFD iteration is compared to FSGS-SFD iteration, the *iter* and *t* have reduced by about 94.69% - 99.49% and 92.72% - 99.45%, respectively. Next, for problem 2, the *iter* and *t* have declined by about 93.12% - 99.17% and 90.48% - 98.93%, respectively, if the HSSOR-SFD iteration is compared to the FSGS-SFD iteration. When FSSOR-SFD iteration is compared to FSGS-SFD iteration, the *iter* and *t*. have reduced by about 90.03% - 98.81% and 86.67% - 98.73%, respectively. Lastly, for problem 3, HSSOR-SFD and FSSOR-SFD iterations need about 86.24% - 98.71% and 78.70% - 97.98%, respectively lesser *iter* when it is compared against FSGS-SFD. While in terms of *t*, HSSOR-SFD and FSSOR-SFD iterations are much faster than FSGS-SFD iterations to complete the computing by about 80.43% - 98.32% and 76.09% - 97.82%, respectively. Overall, the accuracy of the three numerical methods, i.e., HSSOR-SFD, FSSOR-SFD, and FSGS-SFD iterations, are comparable.

Table 1. Performance of the proposed iterations in terms of *iter*, *t* and *err* for Problem 1

	Method	Grid Size				
		64x64	128x128	256x256	512x512	1024x1024
iter	FSGS-SFD	4837	17390	61817	216467	743030
	FSSOR-SFD	257	512	1024	1944	3812
	HSSOR-SFD	160	316	627	1243	2469
t	FSGS-SFD	0.295376	3.435960	46.37301	648.571074	8900.067408

	FSSOR-SFD	0.016110	0.125246	0.921343	6.863014	54.616080
	HSSOR-SFD	0.005352	0.032468	0.246970	2.124093	17.647942
err	FSGS-SFD	2.311010E-07	7.983023E-08	5.370245E-07	2.204651E-06	8.833237E-06
	FSSOR-SFD	2.596319E-07	6.412089E-08	1.493691E-08	7.116531E-09	1.896090E-08
	HSSOR-SFD	1.039366E-06	2.595287E-07	6.414711E-08	1.491863E-08	1.089272E-08



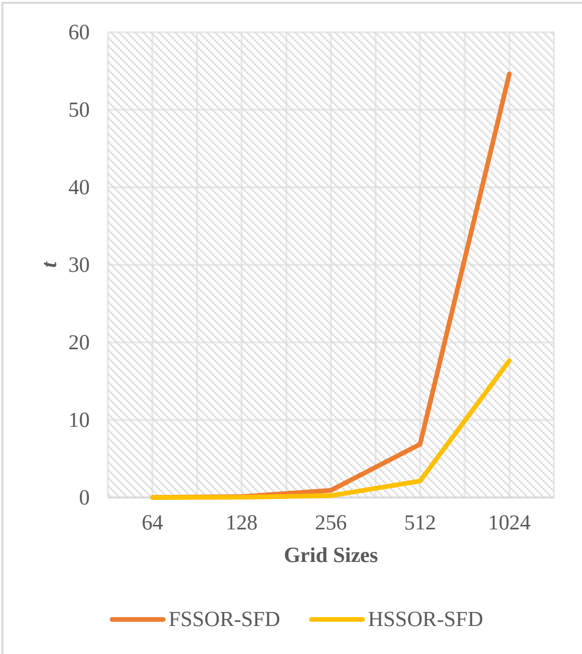


Fig. 5 *iter* and *t* versus Grid Sizes of two different techniques for Problem 1

Table 2. Performance of the proposed iterations in terms of *iter*, *t* and *err* for Problem 2

	Method	Grid Sizes				
		64x64	128x128	256x256	512x512	1024x1024
iter	FSGS-SFD	1584	5476	18590	61010	186434
	FSSOR-SFD	158	308	595	1148	2217
	HSSOR-SFD	109	211	410	800	1556
t	FSGS-SFD	0.112378	1.105432	13.917351	180.885969	2236.354764
	FSSOR-SFD	0.013785	0.074993	0.538216	4.081519	31.306165
	HSSOR-SFD	0.008067	0.022833	0.164887	1.381354	11.171196
err	FSGS-SFD	2.416858E-05	2.415568E-05	2.411302E-05	2.390946E-05	2.294426E-05
	FSSOR-SFD	2.417184E-05	2.416981E-05	2.417365E-05	2.417362E-05	2.417368E-05
	HSSOR-SFD	2.419785E-05	2.417633E-05	2.417530E-05	2.417411E-05	2.417391E-05

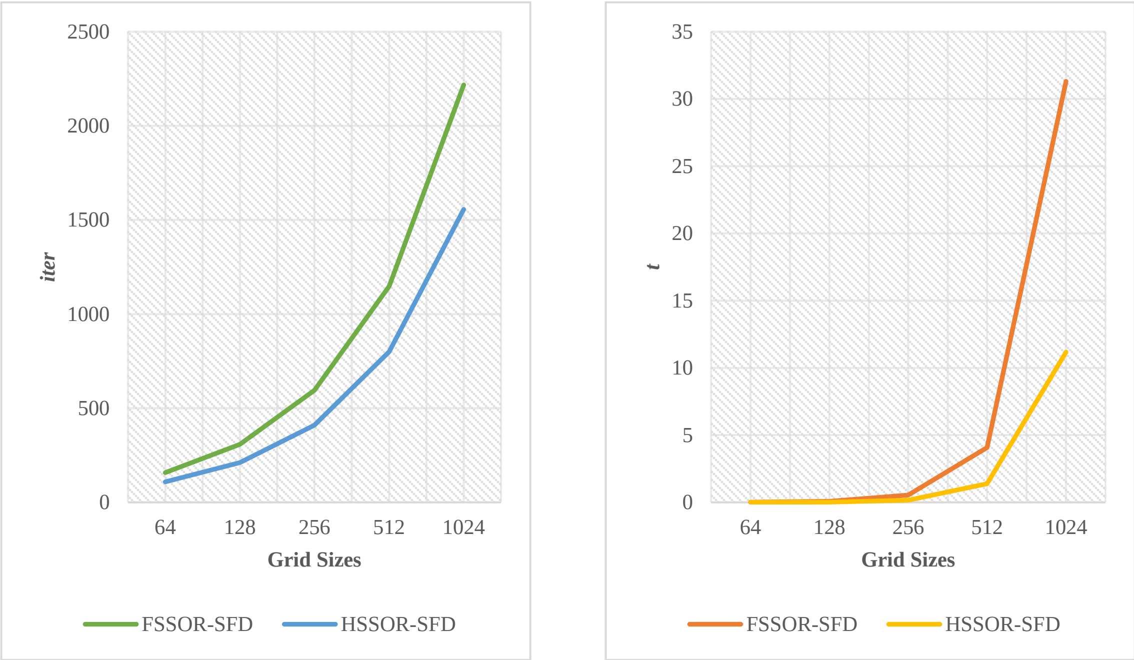


Fig. 6 *iter* and *t* versus Grid Sizes of two different techniques for Problem 2

Table 3. Performance of the proposed iterations in terms of *iter*, *t* and *err* for Problem 3

	Method	Grid Size				
		64x64	128x128	256x256	512x512	1024x1024
iter	FSGS-SFD	676	2412	8645	30801	107678
	FSSOR-SFD	144	286	564	1107	2176
	HSSOR-SFD	93	183	359	709	1398
t	FSGS-SFD	0.043651	0.491499	6.688092	93.608508	1306.817991
	FSSOR-SFD	0.012655	0.068769	0.511474	3.978834	30.994040
	HSSOR-SFD	0.008148	0.020251	0.151031	1.230542	10.124107
err	FSGS-SFD	6.188434E-05	6.189757E-05	6.189644E-05	6.187241E-05	6.173325E-05
	FSSOR-SFD	6.188451E-05	6.189865E-05	6.190241E-05	6.190352E-05	6.190374E-05
	HSSOR-SFD	6.192869E-05	6.190967E-05	6.190516E-05	6.190421E-05	6.190391E-05

Table 4. Percentage reduction in *iter* and *t* of the HSSOR-SFD and FSSOR-SFD iterations in comparison to FSGS-SFD iteration

Problem	Methods	Iteration	Time
1	FSSOR-SFD	94.69% - 99.49%	98.91% - 99.77%
	HSSOR-SFD	96.69% - 99.67%	99.72% - 99.93%
2	FSSOR-SFD	90.03% - 98.81%	98.09% - 99.48%
	HSSOR-SFD	93.12% - 99.17	99.23% - 99.81%
3	FSSOR-SFD	78.70% - 97.98%	96.20% - 99.10%
	HSSOR-SFD	86.24% - 98.70%	98.23% - 99.71%

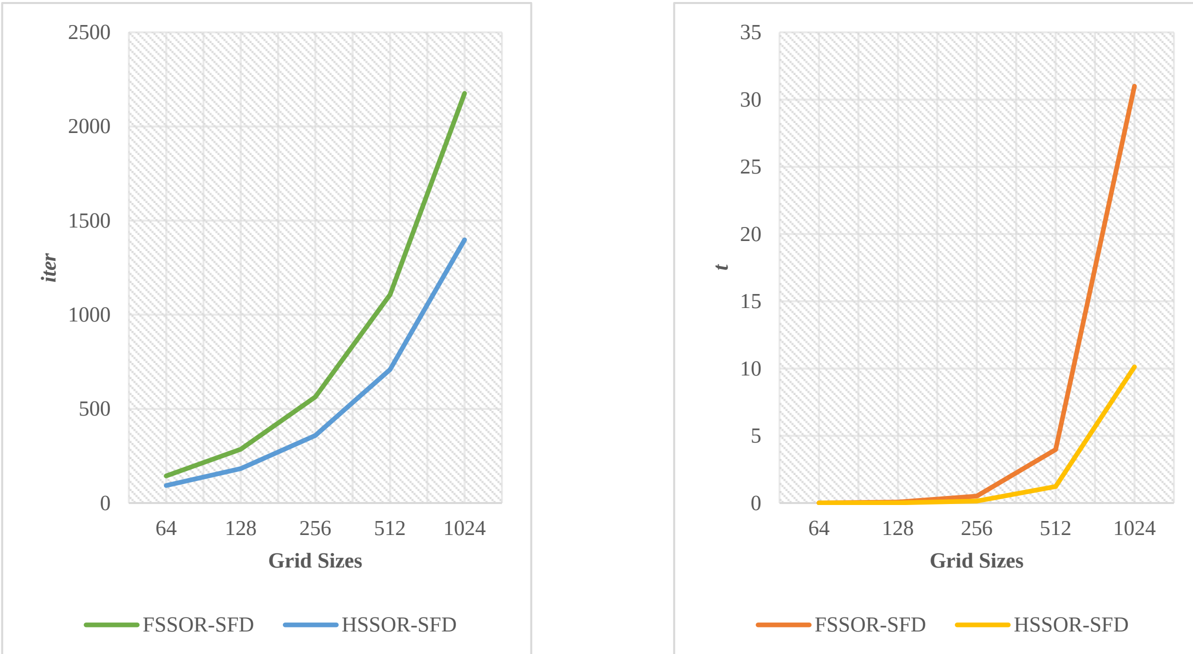


Fig. 7 *iter* and *t* versus Grid Sizes of two different techniques for Problem 3

6. Conclusion

This study described the mathematical derivation of the rotated five-point similarity finite difference approximation equation. This approximation equation leads to a linear system then solved using HSSOR-SFD iteration. We have proven that the *iter* of the HSSOR-SFD iteration is smaller than FSSOR-SFD and FSGS-SFD iterations. Also, we conclude that the rate of convergence of the HSSOR-SFD iteration is faster than the rate of convergence of the FSSOR-SFD and FSGS-SFD iterations. So, we can point out that the HSSOR-SFD iteration is verified to give the best performance in solving two-dimensional hyperbolic telegraph equations. To extend this study, we consider the quarter-sweep iteration as done by [54-56] to speed up the convergence rate of the method.

Funding Statement

The authors received financial support from the Universiti Malaysia Sabah through UMS Great research grant for postgraduate students [GUG0491-1/2020].

Acknowledgements

The authors would like to thank the reviewers for all the useful and helpful comments on our paper. They also gratefully acknowledge the Universiti Malaysia Sabah for funding this research under the UMSGreat research grant for a postgraduate student [GUG0491-1/2020].

References

[1] M. Dehghan, and A. Mohebbi, “The Combination of Collocation, Finite Difference, and Multigrid Methods for Solution of the Two-Dimensional Wave Equation,” *Numerical Methods for Partial Differential Equations: An International Journal*, vol. 24, no. 3, pp. 897-910, 2008.

[2] R. C. Tautz, and I. Lerche, “Application of the Three-Dimensional Telegraph Equation to Cosmic-Ray Transport,” *Research in Astronomy and Astrophysics*, vol. 16, no. 10, pp. 162, 2016.

[3] J. Banasiak, and J.R. Mika, “Singularly Perturbed Telegraph Equations with Applications in the Random Walk Theory,” *Journal of Applied Mathematics and Stochastic Analysis,* vol. 11, no. 1, pp. 9-28, 1998.

[4] V. K. Srivastava, M. K. Awasthi, R. K. Chaurasia, and M. Tamsir, “The Telegraph Equation and its Solution by Reduced Differential Transform Method,” *Modelling and Simulation in Engineering,* vol. 2013, no. 1, pp. 1-6, 2013.

[5] P. M. Jordan, and A. Puri, “Digital Signal Propagation in Dispersive Media,” *Journal of Applied Physics,* vol. 85, no. 3, pp. 1273-1282, 1999.

[6] B. K. Singh, J. P. Shukla, and M. Gupta, “Study of One-Dimensional Hyperbolic Telegraph Equation via a Hybrid Cubic B-Spline Differential Quadrature Method,” *International Journal of Applied and Computational Mathematics,* vol. 7, no. 1, pp. 1-17, 2021.

[7] G. Arora, and V. Joshi, “Comparison of Numerical Solution of 1D Hyperbolic Telegraph Equation using B-Spline and Trigonometric B-Spline by Differential Quadrature Method,” *Indian Journal of Science and Technology*, vol. 9, no. 45, pp. 1-8, 2016.

[8] R. C. Mittal, and R. A. Bhatia, "Numerical Study of Twodimensional Hyperbolic Telegraph Equation by Modified B-Spline Differential Quadrature Method," *Applied Mathematics and Computation*, vol. 244, no. 1, pp. 976-97, 2014.
[9] B. K. Singh, and P. Kumar, "An Algorithm Based on a New DQM with Modified Extended Cubic B-Splines for Numerical Study of Two Dimensional Hyperbolic Telegraph Equation," *Alexandria Engineering Journal*, vol. 57, no. 1, pp. 175-191, 2018.
[10] M. Javidi, "Chebyshev Spectral Collocation Method for Computing Numerical Solution of Telegraph Equation," *Computational Methods for Differential Equations*, vol. 1, no. 1, pp. 16-29, 2013.
[11] S. Jator, "Block Unification Scheme for Elliptic, Telegraph, and Sine-Gordon Partial Differential Equations," *American Journal of Computational Mathematics*, vol. 5, no. 1, pp. 175-185, 2015.
[12] H. Zhou, G. Kong, H. Liu, and L. Laloui, "Similarity Solution for Cavity Expansion in Thermoplastic Soil," *International Journal for Numerical and Analytical Methods in Geomechanics*, vol. 42, no. 2, pp. 274-294, 2018.
[13] A. N. Ceretani, and D. A.Tarzia, "Similarity Solution for a Two-Phase One-Dimensional Stefan Problem with a Convective Boundary Condition and a Mushy Zone Model," *Computational and Applied Mathematics*, vol. 37, no. 2, pp. 2201-2217, 2018.
[14] N.A.M. Ali, J. Sulaiman, A. Saudi, and N.S. Mohamad, "The Similarity Finite Difference Solutions for Two-Dimensional Parabolic Partial Differential Equations via SOR Iteration," *In Computational Science and Technology*, vol. 724, no. 1, pp. 515-526, 2021.
[15] Y. Wang, L. Lin, and J. Li, "A similarity Solution for Undrained Expansion of a Cylindrical Cavity in K 0-Consolidated Anisotropic Soils," *Geomechanics and Engineering*, vol. 25, no. 4, pp. 303-315, 2021.
[16] N.A.M. Ali, R. Rahman, J. Sulaiman, and K. Ghazali, "Numerical Solutions of Unsteady Advection-Diffusion Equations by Using EG Iteration with Wave Variable Transformation," *Journal of Physics: Conference Series*, vol. 1358, pp. 012049, 2019.
[17] F. Ureña, L. Gavete, J.J. Benito, A. García, and A.M. Vargas, "Solving the Telegraph Equation in 2-D and 3-D using Generalized Finite Difference Method (GFDM)," *Engineering Analysis with Boundary Elements*, vol. 112, pp. 13-24, 2020.
[18] M. Aslefallah, and D. Rostamy, "Application of the singular Boundary Method to the Two-Dimensional Telegraph Equation on Arbitrary Domains," *Journal of Engineering Mathematics*, vol. 118, no. 1, pp. 1-14, 2019.
[19] E. Hesameddini, and E. Asadolahifard, "A New Spectral Galerkin Method for Solving the Two Dimensional Hyperbolic Telegraph Equation," *Computers and Mathematics with Applications*, vol. 72, no. 7, pp. 1926-1942, 2016.
[20] V. Devi, R.K. Maurya, S. Singh, and V.K. Singh, "Lagrange's Operational Approach for the Approximate Solution of Two-Dimensional Hyperbolic Telegraph Equation Subject to Dirichlet Boundary Conditions," *Applied Mathematics and Computation*, vol. 367, pp. 124717, 2020.
[21] Y. Zhou, W. Qu, Y. Gu, and H. Gao, "A Hybrid Meshless Method for the Solution of the Second Order Hyperbolic Telegraph Equation in Two Space Dimensions," *Engineering Analysis with Boundary Elements*, vol. 115, pp. 21-27, 2020.
[22] J.J. Benito, F. Urena, and L. Gavete, "Solving Parabolic and Hyperbolic Equations by the Generalized Finite Difference Method," *Journal of Computational and Applied Mathematics*, vol. 209, no. 2, pp. 208-233, 2007.
[23] H. Ding, and Y. Zhang, "A New Fourth-Order Compact Finite Difference Scheme for the Two-Dimensional Second-Order Hyperbolic Equation," *Journal of Computational and Applied Mathematics,"* vol. 230, no. 2, pp. 626-632, 2009.
[24] D. M. Young, "*Iterative Solution of Large Linear Systems*," London: Academic Press, 1971.
[25] W. Hackbusch, "*Iterative Solution of Large Sparse Systems of Equations*," Springer-Verlag, 1995.
[26] Y. Saad, "*Iterative Methods for Sparse Linear Systems*," International Thomas Publishing, 1996.
[27] A. Saudi, and J. Sulaiman, "Half-Sweep Gauss-Seidel (HSGS) Iterative Method for Robot Path Planning," In the 3rd *Int. Conf. on Informatics and Technology (Informatics09)*, Kuala Lumpur, pp. 27-28, 2009.
[28] L. H. Ali, J. Sulaiman, and S.R.H. Hashim, "Numerical Solution of Fuzzy Fredholm Integral Equations of Second Kind using Half-Sweep Gauss-Seidel Iteration," *Journal of Engineering Science and Technology*, vol. 15, pp. 3303-3313, 2020.
[29] N.A.M. Ali, R. Rahman, J. Sulaiman, and K. Ghazali, "Solutions of Reaction-Diffusion Equations using Similarity Reduction and HSSOR Iteration," *Indonesian Journal of Electrical Engineering and Computer Science*, vol. 16, pp. 1430-1438, 2019.
[30] N. S.Mohamad,J. Sulaiman, A.Saudi, and N.F.A. Zainal, "First Order Piecewise Collocation Solution of Fredholm Integral Equation Second Type using Sor Iteration," In *Computational Science and Technology, Singapore: Springer*, pp. 395-406, 2021.
[31] R. Rahman, N.A.M Ali, J. Sulaiman, and F. Muhiddin, "Caputo's Finite Difference Solution of Fractional Two-Point Boundary Value Problems using SOR Iteration," In *AIP Conference Proceedings*, vol. 2013, no. 1, pp. 020034, 2018.
[32] M. M. Xu, J. Sulaiman, and L. H. Ali, "Linear Rational Finite Difference Approximation for Second-Order Linear Fredholm Integro-Differential Equations using the Half-Sweep SOR Iterative Method," *International Journal of Engineering Trends and Technology*, vol. 69, no. 6, pp. 136-143, 2021.
[33] N.Z.F.M. Radzuan, M.N. Suardi, and J. Sulaiman, "KSOR Iterative Method with Quadrature Scheme for Solving System of Fredholm Integral Equations of Second Kind," *Journal of Fundamental and Applied Sciences*, vol. 9, no, 5S, pp. 609-623, 2017.
[34] F.A. Muhiddin, J. Sulaiman, and A. Sunarto, "A Class of Weighted Point Schemes for the Grünwald Implicit Finite Difference Solution of Time-Fractional Parabolic Equations using KSOR Method," In *Journal of Physics: Conference Series*, vol. 1298, no. 1, pp. 012001, 2019.
[35] A.A. Dahalan, A. Saudi, and J. Sulaiman, "Path Finding of Static Indoor Mobile Robot via AOR Iterative Method Using Harmonic Potentials," *ZULFAQAR International Journal of Defence Science, Engineering & Technology*, vol. 3, no. 2, pp. 18-24, 2020.
[36] A.A. Dahalan, M. S. Muthuvalu, and J. Sulaiman, "Numerical Solutions of Two-Point Fuzzy Boundary Value Problem using Half-Sweep Alternating Group Explicit Method," *American Institute of Physics*, vol. 1557, no. 1, pp. 103-107, 2013.
[37] A.A. Dahalan, M.S. Muthuvalu, and J. Sulaiman, "Numerical Solutions of Two-Point Fuzzy Boundary Value Problem using Half-Sweep Alternating Group Explicit Method," In *AIP Conference Proceedings*, vol. 1557, no. 1, pp. 103-107, 2013.
[38] A.R. Abdullah, "The four Point Explicit Decoupled Group (EDG) Method: A Fast Poisson Solver," *International Journal of Computer Mathematics*, vol. 38, no. 1-2, pp. 61-70, 1991.

[39] M.K. Hasan, M. Othman, Z. Abbas, J. Sulaiman, and F. Ahmad, "Parallel Solution of High Speed Low Order FDTD on 2D Free Space Wave Propagation," In *International Conference on Computational Science and Its Applications*, Berlin, Heidelberg: Springer, pp. 13-24, 2007.

[40] M. K. M. Akhir, M. Othman, J. Sulaiman, Z.A. Majid, and M. Suleiman, "The Four Point-EDGMSOR Iterative Method for Solution of 2D Helmholtz Equations," In *International Conference on Informatics Engineering and Information Science,* Berlin, Heidelberg:Springer, pp. 218-227, 2011.

[41] R. Rahman, N.A.M Ali, J. Sulaiman, and F. Muhiddin, "Application of the Half-Sweep EGSOR Iteration for Two-Point Boundary Value Problems of Fractional Order," *Advances in Science, Technology and Engineering Systems,* vol. 4, no. 2, pp. 237-243, 2019.

[42] N. F. A. Zainal, J. Sulaiman, and M.U. Alibubin, "Application of Half-Sweep SOR Iteration with Nonlocal Arithmetic Discretization Scheme for Solving Burgers' Equation," *ARPN J. Eng. Appl. Sci*., vol. 14, pp. 616-621, 2019.

[43] M. K. M. Akhir, M. Othman, J. Sulaiman, Z. A. Majid,andM. Suleiman, "Half-sweep Modified Successive Overrelaxation for Solving Two-Dimensional Helmholtz Equations," *Australian Journal of Basic and Applied Sciences*, vol. 5, no. 12, pp. 3033-3039, 2011.

[44] N.A.M. Ali, J. Sulaiman, A. Saudi, and N.S. Mohamad, "SOR Iterative Method with Wave Variable Transformation for Solving Advection-Diffusion Equations," *AIP Conference Proceeding,* vol. 2013, pp. 020036, 2018.

[45] N.A.M. Ali, J. Sulaiman, A. Saudi, and N.S. Mohamad, "Performance of Similarity Explicit Group Iteration for Solving 2D Unsteady Convection-Diffusion Equation," *Indonesian Journal of Electrical Engineering and Computer Science,* vol. 23, no. 1, pp. 472-478, 2021.

[46] D.M. Young, "*Iterative Solution of Large Linear Systems*," London: Academic Press, 1971.

[47] Y. Saad, "*Iterative Methods for Sparse Linear Systems*," 2nd edn. Philadelphia, PA: SIAM, 2003.

[48] J. Sulaiman, M.K. Hasan, M. Othman, and S.A.A. Karim, "Fourth-Order Solutions of Nonlinear Two-Point Boundary Value Problems by Newton-HSSOR Iteration," *AIP Conference Proceeding,* vol. 1602, pp. 69-75, 2014.

[49] A.M. Saeed and N.H.M. Ali, "On the Convergence of the Preconditioned Group Rotated Iterative Methods in the Solution of Elliptic PDEs," *Applied Mathematics and Information Sciences,* vol. 5, no. 1, pp. 65-73, 2011.

[50] A. Ibrahim and A. R. Abdullah, "Solving the Two Dimensional Diffusion Equation by the Four Point Explicit Decoupled Group (EDG) Iterative Method," *International Journal of Computer Mathematics*, vol. 58, pp. 253-263, 1994.

[51] J. Lin, F. Chen, Y. Zhang, and J. Lu, "An Accurate Meshless Collocation Technique for Solving Two-Dimensional Hyperbolic Telegraph Equations in Arbitrary Domains," *Engineering Analysis with Boundary Elements,* vol. 108, pp. 372-384, 2019.

[52] J. Lin, F. Chen, Y. Zhang, and J. Lu, "An Accurate Meshless Collocation Technique for Solving Two-Dimensional Hyperbolic Telegraph Equations in Arbitrary Domains," *Engineering Analysis with Boundary Elements,* vol. 108, pp. 372-384, 2019.

[53] B.K. Singh and P. Kumar, "An Algorithm Based on a New DQM with Modified Extended Cubic B-Splines for Numerical Study of Two Dimensional Hyperbolic Telegraph Equation," *Alexandria Engineering Journal,* vol. 57, no. 1, pp. 175-191, 2018.

[54] A. Sunarto, and J. Sulaiman, "Implementation QSGS Iteration Applied to Fractional Diffusion Equation," In *Journal of Physics: Conference Series,* vol. 1363, no. 1, pp. 012086, 2019.

[55] J. Sulaiman, M. Othman, and M.K. Hasan, "Quarter-Sweep Iterative Alternating Decomposition Explicit Algorithm Applied to Diffusion Equations," *International Journal of Computer Mathematics,* vol. 81, no. 12, pp. 1559-1565, 2004.

[56] J.V. Lung, and J. Sulaiman, "On Quarter-Sweep Finite Difference Scheme for One-Dimensional Porous Medium Equations," *International Journal of Applied Mathematics,* vol. 33, no. 3, pp. 439, 2020.