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Original Article

Application of Refinement Successive Over-Relaxation (RSOR) in Solving the Piecewise Polynomial on Fredholm Integral Equation of the Second Type

Nor Syahida Mohamad[1], Jumat Sulaiman[2], Azali Saudi[3], Nur Farah Azira Zainal[4]

[1,2,4]*Faculty of Science and Natural Resources, Universiti Malaysia Sabah, Kota Kinabalu, Sabah Malaysia.*
[3]*Faculty of Computing and Informatics, Universiti Malaysia Sabah, Kota Kinabalu, Sabah, Malaysia.*

[1]*Corresponding Author : norsyahida1302@gmail.com*

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Abstract *- This paper establishes an effective and reliable algorithm for solving the second type of FIE based on the first-order piecewise polynomial and the first-order quadrature method. The algorithm, which is called Composite Trapezium (CT), is generally used to discretize any integral term. This paper also aims to derive a Composite Trapezium (CT) with first-order piecewise polynomial and first-order quadrature linear collocation approximation equation generated from the discretization process of the proposed problem by considering the distribution of node points with vertex-centered. Accordingly, we built a system of CT linear collocation approximation equations using collocation node points over the approximation equation for linear collocation. The coefficient matrix is large and dense. In addition, this research also considered the effective Refinement Successive Over-Relaxation (RSOR) algorithm to obtain the piecewise linear collocation solution of this linear problem. In order to test the proposed iterative methods, three tested examples were solved. The results were subsequently obtained based on three parameters, including the iterations (I), execution period (s), and the maximum absolute error, which was all recorded and further compared with two iterations, SOR and RSOR.*

Keywords *- Piecewise, Collocation, Successive Over-Relaxation (SOR) method, and Refinement Successive Over-Relaxation (RSOR) method.*

1. Introduction

Integration has a concept in which an integral equation is structured as an equation where it can be seen as an unknown function $u(x)$ that needs to be resolved under the integral sign [1]. Based on the historical background, the first produced integral equation comes from a curve graph with heavy particles connected and sliding down in descending order without any friction to the lowest position (see Figure 1). In this case, the curve graph can form many equations, especially in potential and kinetic energy; thus, the evolution of science will lead to the new formation of physical laws over time and will frequently appear in many fields such as engineering, quantum mechanics, medical, image, and others [2, 3, 4]. The integral equation is also a beneficial tool in many areas of study, and it has huge applications in most physical problems. Such problems are applicable in image, engineering, and mechanic quantum fields as a main idea in the studies. The implications of the integration in those fields are commonly used to solve and improve the efficiency of numerical results. One of the examples is its implication in Electrocardiographic Imaging to improve the efficiency of numerical results [5]. However, there are several classifications of the integral equation, which can include linear and nonlinear integral, and the frequently used integral equations are Volterra, Fredholm, integro-differential, and singular integral equations.

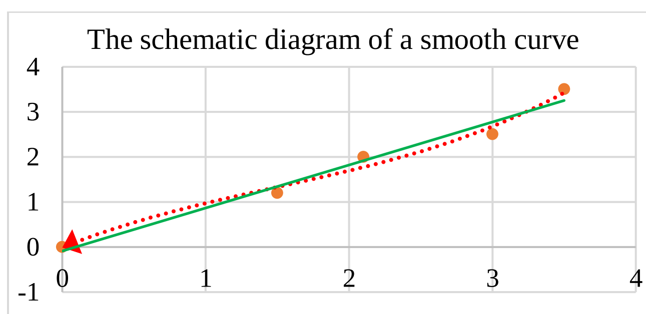

Fig. 1 Integral curve graph

The following entails how these four major types of integral equations can be distinguished:

Volterra integral equations [6]:

$$\vartheta(x)u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt \quad (1.1)$$

Fredholm integral equations [7]:

$$\vartheta(x)u(x) = g(x) \quad (1.2)$$

Singular integral equations [8]:

$$u(x) = f(x) + \lambda \int_{-\infty}^{\infty} u(t)dt$$

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha}.u(t)dt, 0 < \alpha < 1 \quad (1.3)$$

Integro-differential equations [9]:

$$L\frac{dI}{dt} + RI + \frac{1}{c}\int_0^t I(\tau)d\tau = f(t), I(0) = I_0 \quad (1.4)$$

L = Inductance, R = Resistance, C = Capacitance

In reference to Eq. (1.2), the original formula of Fredholm integral equations denotes the third type of equation. If the function is $\vartheta(x) = 1,$ then (1.2) will turn into

$$u(x) + g(x) + \lambda \int_a^b K(x,t)u(t) \quad (1.21)$$

and the equations named Fredholm integral equation of the second type when $\vartheta(x) = 0$ then (1.21) yields

$$g(x) + \lambda \int_a^b K(x,t)u(t)\, dt = 0, \quad (1.22)$$

which is named the Fredholm integral equation of the first type.

2. Research Gap

Fredholm integral equation of the second type (1.21) is the main problem in this study. As the study aims to obtain the approximate equations, several methods were listed, such as the Galerkin method, the Cauchy method, the B-Spline method, and many more [10, 11, 12]. However, previous studies have shown that the collocation method is simple and easy for generating the network grid on the domain solutions [13]. Moreover, the collocation method has less complexity compared to other methods, such as the Galerkin method. Therefore, this study has selected the most reliable and suitable approach to obtain the approximation equations. Besides, this study was also inspired by the most recent studies where the quadrature methods of Newton Cotes rules have been applied in various cases, particularly to obtain the approximation equations and produce the corresponding linear system [14]. In line with the recent studies mentioned, operating the linear system using the quadrature method has proven excellent efficiency in results, and it can be stated that the results showed good agreement.

Therefore, this study focuses on solving FIE of the second type by applying first-degree piecewise polynomial and the first-degree quadrature scheme of the Trapezium method to obtain the approximate equations by imposing the collocation method in the discretization process to generate the corresponding linear system.

Since the smooth kernel FIE of the second type is the main problem highlighted in this study, let the characteristic of Eq. (1.21) be described where

$u(x)$ = an unknown function
$g(x)$ = the provided function
λ = lambda parameter.

3. Discretization Process of Piecewise Polynomial Collocation on FIE of the Second Type

3.1. Discretize

Based on the study by [15], the Trapezium method is one of the first-order quadrature methods of Newton Cotes, which is involved with the two points.

3.1.1. Trapezium Rule

Basic formula:

$$I = h(y_0 + \frac{\Delta y_0}{2}) = h(\frac{y_0+y_1}{2}) = \frac{h}{2}[f(a) + f(b)] \quad (2.0)$$

n-equal interval, such that

$$x_0 = a,$$
$$x_1 = a + h,$$
$$x_2 = a + 2h,$$
$$x_n = a + nh = b.$$

$$I = (\frac{h}{2})[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})] \quad (2.1)$$

$$I = (\frac{h}{2})[(f(a) + f(b)) + 2(f(x_1) + f(x_2) + \cdots + f(x_{n-1})] \quad (2.2)$$

$$\int_a^b f(x)dx = \frac{h}{2}(f_0 + 2\sum_{j=0}^{n} f_i + f_n) \quad (2.3)$$

The above figure is called Trapezium rule as it approximates the small curve part of the function by a straight line and interprets the area under the curved part as the area of the Trapezium, as shown in Fig. 2. The basic formula of the quadrature method will be applied in the discretization process on Fredholm integral equation of the second type. The quadrature method will then take part in integration calculation to speed up the iteration process. This will be further elaborated on, step by step, in the next section.

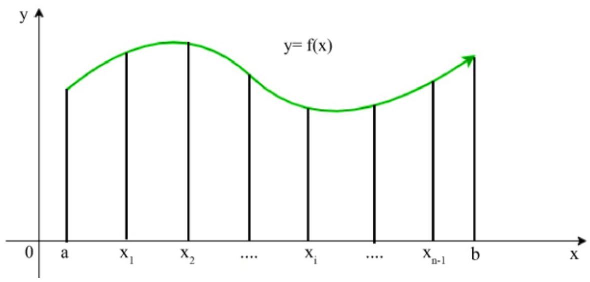

Fig. 2 Geometrical interpretation of Trapezium Rule

In this part, the study introduced the domain solutions of integral $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$, which is uniformly divided with n-subintervals where the node points of the domain solutions with $\mathrm{x_i}$ $\mathrm{i} = ,1,2,3, \dots, \mathrm{n}, \mathrm{n}+1$ can be viewed as

$$a = x_1, x_2 < \cdots < x_n < x_{n+1} = b.$$

The definition of $\boldsymbol{h}$ cannot be neglected as it is always needed to calculate the length size of the subinterval of $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$. The formula of $\boldsymbol{h}$ is depicted in Eq. (2.4); for a better view, see Fig. 3, the domain solutions of I.

$$\mathrm{h} = \frac{\text{b-a}}{\text{n}} \tag{2.4}$$

Fig. 3 Interval domain of $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$ on Fredholm integral equation of the second type.

To describe Fig. 3, the domain solutions denote an edge-vertex type with a first case, which entails a full-sweep network in the MATLAB version. All the node points were the approximate points to be counted in the iteration process later. Consider the distribution for full-sweep node points.

$$x_i = a + ih, i = ,1,2,3, \dots, n, n+1. \tag{2.5}$$

To solve Eq. (1.21), we must perform the discretization process, which includes first-degree polynomials piecewise. Systematically, the first-degree polynomial piecewise approximation will be generated first before it further combines the first-degree quadrature method and the collocation scheme on the domain solutions. The following equation outlines the first-degree polynomial piecewise approximation function of the full-sweep case:

$$U(t) = \sum_{i=1}^{n} H_i(x). \delta_i(x), x \in [a, b], \tag{2.6}$$

The process of discretization starts by applying Eq. (2.6) to Eq. (1.21) in order to generate the corresponding approximation equations, as shown in Eq. (2.7):

$$U(x) + \lambda \int_a^b k(x,t) \sum_{i=1}^{n} R_i(x). \delta_i(x) dt = g(x), \tag{2.7}$$

By expanding Eq. (2.7), we obtain the following equations:

$$\Rightarrow U(x) + \lambda \int_a^b k(x,t) R_1(t). \delta_1(t) dt + R_2(t). \delta_2(t) dt + \cdots + R_n(t). \delta_n(t) dt = g(x), \tag{2.8}$$

$$\Rightarrow U(x) + \lambda \int_a^b k(x,t) R_1(t). \delta_1(t) dt + \lambda \int_a^b k(x,t) R_2(t). \delta_2(t) dt + \cdots + \lambda \int_a^b k(x,t) R_n(t). \delta_n(t) dt = g(x), \tag{2.9}$$

$$\because a = x_1, b = x_{n+1}$$

$$\Rightarrow U(x) + \lambda \int_{x_1}^{x_{n+1}} k(x,t) R_1(t). \delta_1(t) dt + \lambda \int_{x_1}^{x_{n+1}} k(x,t) R_2(t). \delta_2(t) dt + \cdots + \lambda \int_{x_1}^{x_{n+1}} k(x,t) R_n(t). \delta_n(t) dt = g(x), \quad \dots (2.10)$$

$\delta_i(t)$ is defined as

$$\delta_i(x) = \begin{cases} 1, x_{i-1} < x < x_i \\ 0, others \end{cases}, i = 2,3 \cdots, n, n+1.$$

Thus, the new equation satisfies the piecewise constant function; refer to Eq. (2.11):

$$\Rightarrow U(x) + \lambda \int_{x_1}^{x_2} k(x,t) R_1(t). \delta_1(t) dt + \lambda \int_{x_2}^{x_3} k(x,t) R_2(t). \delta_2(t) dt + \cdots + \lambda \int_{x_1}^{x_{n+1}} k(x,t) R_n(t). \delta_n(t) dt = g(x), \tag{2.11}$$

$$\because H_p(x_i) \int_{x_p}^{x_{p+1}} k(x_i,t) R_p(t) dt,$$
$$p = 1,2, \dots, n, \; i = 1,2, \dots, n+1.$$

Let H be in Eq. (2.12) before detailing each part, as we will apply the first-degree polynomial piecewise in this part

$$H_p(x_i) \int_{x_p}^{x_{p+1}} k(x_i,t) R_p(t) dt,$$
$$p = 1,2, \dots, n, \; i = 1,2, \dots, n+1. \tag{2.12}$$

The first-degree polynomial piecewise is defined below.

$$\therefore R_p(t) = \left(\frac{x_{p+1} - t}{h}\right) u(x_p) + \frac{t - x_p}{h} u(x_{p+1})$$

$$\therefore H_p(x_i) = \int_{x_p}^{x_{p+1}} k(x_i,t)[(\frac{x_{p+1}\text{-}t}{h})u(x_p) + (\frac{t\_x_p}{h})u(x_{p+1})]dt$$
$$= [\frac{1}{h}\int_{x_p}^{x_{p+1}} k(x_i,t)(x_{p+1}\text{-}t)dt]u(x_p) +$$
$$[\frac{1}{h}\int_{x_p}^{x_{p+1}} k(x_i,t)(t\text{-}x_p)dt]u\,(x_{p+1}) \quad (2.13)$$

Before that, the application of composite Trapezium rule will be applied to the integral part where it is used in an iteration process. Based on Eq. (2.3), the coefficient of $A_j, j = 1,2,3,4,\dots,n$ in the Trapezium, the rule can be defined as

$$A_j = \begin{cases} h, j = 0, n \\ 2h, others \end{cases}$$

The discretization process is continued by applying the collocation method to all the node points in the approximation equations. Finally, Eq (1.21) will generate a linear system equation of

$$G\underline{U} = \underline{g} \quad (2.14)$$

where

$$G = \begin{bmatrix} 1 + G(x_1, 1) & \cdots & G(x_1, n+1) \\ \vdots & \ddots & \vdots \\ G(x_{n+1}) & \cdots & 1 + G(x_{n+1}, n+1) \end{bmatrix}_{(N+1)(N+1)}$$

$$\underline{U} = \begin{bmatrix} U(x_1) \\ U(x_2) \\ \vdots \\ U(x_{n+1}) \end{bmatrix}_{(N+1)\times 1} \quad \underline{g} = \begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_{n+1}) \end{bmatrix}_{(N+1)\times 1}$$

Resultantly, the linear system shows that it has a huge-scale and dense matrix after applying first-degree polynomial piecewise on Fredholm integration of the second type.

4. Derivative Method : Modification of Relaxation Iterative Method (SOR) into Refinement Successive Over (RSOR)

This section focuses on obtaining the numerical solution of the linear system (2.14), which is generated as a result of the discretization procedure used on the given problem. In general, there are direct approaches and iterative methods that can be taken into consideration to solve the linear system. Nonetheless, it is clear that the coefficient matrix characteristics are massive and dense. As a result, a family of iterative approaches must be taken into account while developing an acceptable solution for such a linear system.

The Gauss-Seidel method is one of the often-used classic iterative methods, as the algorithm itself was inspired by the Jacobi iterative method. Besides, the Gauss-Seidel method has better performance than the Jacobi method. However, Gauss-Seidel is slower than the Successive Over-Relaxation (SOR) method since SOR has a relaxation factor in the algorithm [17, 18, 19, 30]. Consider the linear system of

$$G\underline{U} = \underline{g}$$

$$G \in Max_{n\times n}, g \in \Re^n$$

and let

$$G = D - L - U$$

be the decomposition of G, with Das as the diagonal matrix, L as lower triangular, and U as the upper triangular matrix.

However, in view of the matter of high-speed convergence test, this study considers the RSOR iterative method as the best method among those presented methods in this study [20]. The following equation shows the SOR iterative method that was derived from the linear system:

$$U^{(k+1)} = (D - \omega L)^{-1}[(1-\omega)D + \omega U]U^{(k)} + \omega(D - \omega L)^{-1}g \quad (3.0)$$

Based on a recent study, the application of the refinement theory into the iterative method of SOR has modified the equation to have a better algorithm since it has a faster convergence speed than SOR in many applications [21]. Eq. (3.1) shows the RSOR iterative method where $\boldsymbol{k}$ is the number of iterations and $\boldsymbol{\omega} \in (\mathbf{0}, \mathbf{2})$ is the relaxation factor:

$$U^{(k+1)} = \{(D - \omega L)^{-1}[(1-\omega)D + \omega U]\}U^{(k)} + \omega\{I + (D-\omega L)^{-1}[(1-\omega)D + \omega U)]\}(D-\omega L)^{-1}g \quad (3.1)$$

Algorithm of the RSOR Iterative Method

a. Set the initial value, $\delta = 10^{-10}, U^0 = 0, k = 0.$
b. For $k = 0,1,\dots n$ Calculate.
   I. $U^{(k+1)} = \{(D - \omega L)^{-1}[(1-\omega)D + \omega U]\}U^{(k)} + \omega\{I + (D-\omega L)^{-1}[(1-\omega)D + \omega U)]\}(D - \omega L)^{-1}g$
   II. Do the convergence test. If $\left\|U^{(k+1)}\text{-}U^{k)}\right\|_\infty \leq \delta = 10^{-10}$ is satisfied, then go to step c. Otherwise, go to step b.
c. Display the numerical solutions.

5. Numerical Example

5.1. Problems

This part theoretically explains the process of solving the numerical solution of the Fredholm integral equation of the second type. Three examples were proposed to test the effectiveness of the presented methods, which include the

Successive Over-Relaxation (SOR) method and the Refinement Successive Over-Relaxation (RSOR) method. These methods have been applied in this experiment by considering the optimal parameter of ω as the optimal point for SOR and RSOR. The experiment was executed by using MATLAB software. This study has carried out the numerical experiment, and all results have been tabulated with three parameters such as iterations, execution period, and maximum absolute error with five sizes of mesh size starting with $2^n, n = 8,9,10,11,12$. The results are shown in Table 1 to Table 3. In the process of gaining all the results, this study considered the most crucial part, which is the optimal value needed to meet all the characteristics of the convergence test in reference to Eq. (4.0). The definition of any experiment is aimed at possibly meeting the convergence test where every $e^{(k)}$ must approach zero.

Theorem [22]

$$e^{(k)} = \max s \left|x^{(k+1)}\text{-}x^{(k)}\right| < \varepsilon \qquad (4.0)$$

In addition, to test the efficiency, one of the iterative methods, the SOR method, was set as the control method to record the reduction percentages of iterations and execution periods in all the examples tested in this experiment. Hence, the formula is shown in Eq. (4.1) [31]

$$\Omega = \tau \times 100 \qquad (4.1)$$

where

$$\tau = \frac{SOR - RSOR}{SOR}$$

In reference to Eq. (4.1), the formula will be applied in three examples, as follows:

Example 1 [24]:

$$y(x) = x + \int_0^1 4xt\text{-}x^2 y(t)dt \qquad (4.2)$$

The exact solution of (4.2) is given as

$$y(x) = 24x\text{-}9x^2..$$

Example 2 [25]:

$$y(x) = x + \int_0^1 (xt^2 + tx^2)y(x)dt, \qquad (4.3)$$

The exact solution of (4.3) is given as

$$y(x) = \frac{80}{119}x^2 + \frac{180}{119}x.$$

Example 3 [26]:

$$y(x) = \sin(2\pi x) + \int_0^1 \cos(x)ydt, \qquad (4.4)$$

The exact solution of (4.4) is given as

$$y(x) = \sin(2\boldsymbol{\pi} x).$$

5.2. Results

Table 1. Iterations for three examples of FIE second type

Example	M	Iteration (I)	
		SOR	RSOR
1	256	43 (w=1.546)	22 (w=1.559)
	512	44 (w=1.553)	22(w=1.555)
	1024	44 (w=1.551)	22(w=1.551)
	2048	45 (w=1.552)	23 (w=1.551)
	4096	45 (w=1.551)	23(w=1.551)
2	256	14 (w=1.121)	8 (w=1.141)
	512	14 (w=1.121)	8 (w=1.141)
	1024	14 (w=1.121)	8 (w=1.142)
	2048	14 (w=1.121)	8 (w=1.147)
	4096	14 (w=1.121)	8 (w=1.111)
3	256	27 (w=1.361)	14 (w=1.366)
	512	27 (w=1.361)	14 (w=1.366)
	1024	28 (w=1.361)	14(w=1.361)
	2048	28 (w=1.361)	15(w=1.352)
	4096	28 (w=1.361)	15(w=1.359)

Table 2. Execution periods for three examples of FIE second type

Example	n	Execution period (s)	
		SOR	RSOR
1	256	0.5657	0.3059
	512	2.5425	1.4730
	1024	10.5422	7.6244
	2048	46.4116	41.3226
	4096	252.4253	247.5704
2	256	0.5990	0.5538
	512	2.5568	2.5482
	1024	10.5866	10.4701
	2048	46.5324	46.4234
	4096	262.1220	249.2334
3	256	0.6360	0.3544
	512	2.7732	1.6414
	1024	8.9257	8.4417
	2048	49.6225	44.2933
	4096	275.8081	261.1481

Table 3. Max. Abs. Error for three examples on FIE second type

Example	M	Max. abs. Error (Max. R)	
		SOR	RSOR
1	256	3.96E-04	3.96E-04
	512	9.90E-05	9.90E-05
	1024	2.48E-05	2.48E-05
	2048	6.19E-06	6.19E-06
	4096	1.55E-06	1.55E-06
2	256	3.09E-06	3.09E-06
	512	7.72E-07	7.72E-07
	1024	1.93E-07	1.93E-07
	2048	4.82E-08	4.82E-08
	4096	1.20E-08	1.21E-08
3	256	1.68E-12	1.09E-12
	512	1.70E-12	1.04E-12
	1024	9.16E-13	9.19E-13
	2048	9.20E-13	2.57E-13
	4096	9.21E-13	1.27E-13

Table 4. Reduction percentages for three examples on FIE second type

Example	Iterations (I)%	Execution period (s)%
1	48.84 - 51.11	10.96 - 45.93
2	42.86	0.23 - 7.56
3	46.43 - 50.00	5.32 - 44.28

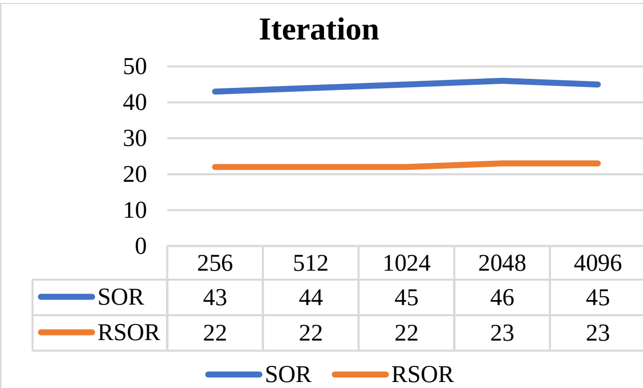

Fig. 4 Iteration over Problem 1

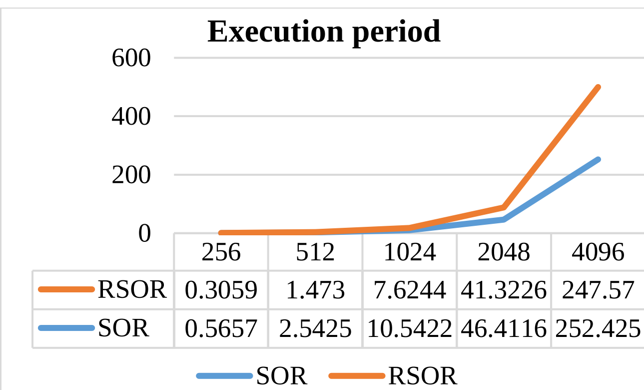

Fig. 5 Execution period over problem 1

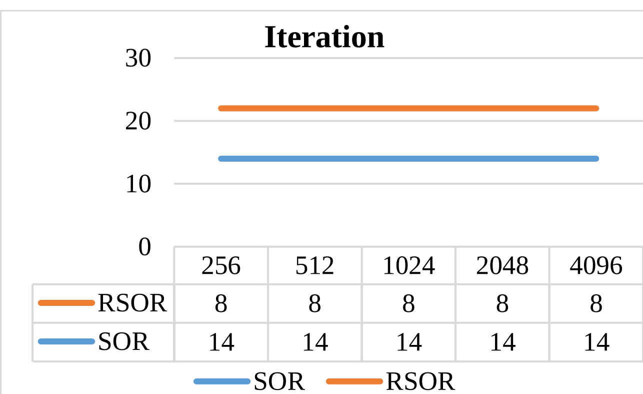

Fig. 6 Iteration over Problem 2

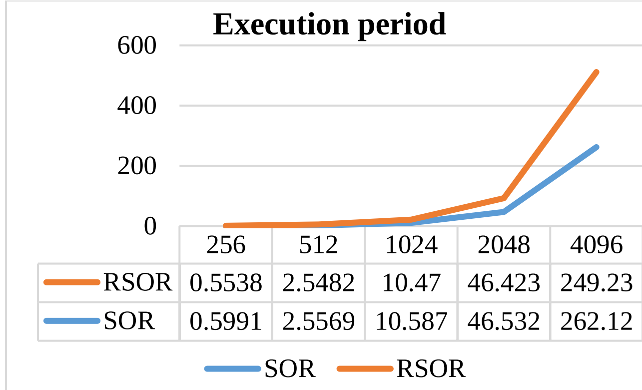

Fig. 7 Execution period over problem 2

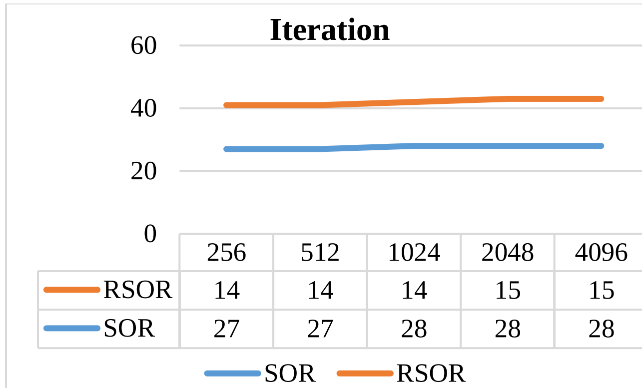

Fig. 8 Iteration over Problem 3

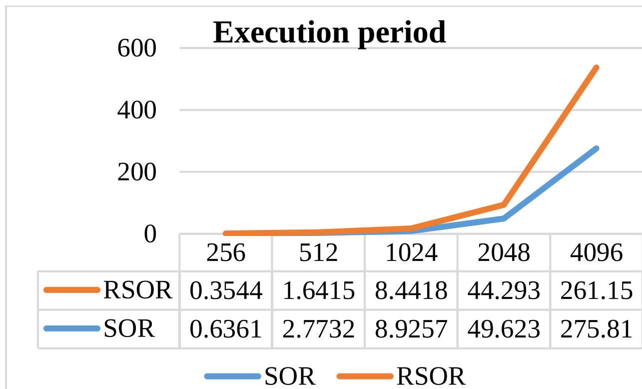

Fig. 9 Execution period over problem 3

5.3. Discussion

Based on Table IV, there are significant differences in iterations (I) and execution period (s). The formula implied in the current study has portrayed that the reduction percentages of iterations (I) and execution period (s) recorded from the RSOR iterative method were 48.84%-51.11%, 42.86%, and 46.43%-50.00%, respectively. Meanwhile, the execution period (s) constituted 10.96%-45.93%, 0.23%-7.56%, and 5.32%-44.28%, respectively. These numerical computational results on the SOR and RSOR family in Tables 1-3 showed that the RSOR iterative method had small iterations (I) and execution period (s) compared to the RSOR iterative method due to the modification of its algorithms.

6. Conclusion

The conclusion drawn from numerical computations on FIE of the second type with first-degree polynomial piecewise and a combination of first-degree quadrature with the RSOR iterative method is that it is superior to SOR in terms of iterations (I) and execution period (s) due to lower operational complexity because of the modification and implication of the refinement theory on SOR formula, which was modified to increase the rate of convergence of the iteration process. Thus, the RSOR iterative method is better than the SOR iterative method. Overall, future works can be discussed in a higher-order quadrature scheme, where we can expand the Fredholm integral of the second type in the closed Newton-Cotes family [27, 28]. Future works can also involve the new modification of the RSOR family [25, 29].

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References

[1] H. Hofer, K. Wysocki, and E. Zehnder, "Integration Theory on the Zero Sets of Polyfold Fredholm Sections," *Mathematische Annalen*, vol. 346, pp. 139–198, 2010. [CrossRef] [Google Scholar] [Publisher Link]

[2] Thomas Konrad, and Andrew Forbes, "Quantum Mechanics and Classical Light," *Contemporary Physics*, vol. 60, no. 1, pp. 1-22, 2019. [CrossRef] [Google Scholar] [Publisher Link]

[3] Aceng Sambas et al., "A 3-D Multi-Stable System with a Peanut-Shaped Equilibrium Curve: Circuit Design, FPGA Realization, and an Application to Image Encryption," *IEEE Access*, vol. 8, pp. 137116-137132, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[4] Pedro Ivan Tello Flores, "Approach of RSOR Algorithm Using HSV Color Model for Nude Detection in Digital Images," *Computer and Information Science*, vol. 4, no. 4, pp. 29-45, 2011. [CrossRef] [Google Scholar] [Publisher Link]

[5] Subham Ghosh, and Yoram Rudy, "Accuracy of Quadratic Versus Linear Interpolation in Noninvasive Electrocardiographic Imaging (ECGI)," *Annals of Biomedical Engineering*, vol. 33, pp. 1187–1201, 2005. [CrossRef] [Google Scholar] [Publisher Link]

[6] Sumati Kumari Panda, Erdal Karapınar, and Abdon Atangana, "A Numerical Schemes and Comparisons for Fixed Point Results with Applications to the Solutions of Volterra Integral Equations in Dislocated Extended b- Metric Space," *Alexandria Engineering Journal*, vol. 59, no. 2, pp. 815-827, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[7] N.S. Mohamad, and J. Sulaiman, "The Piecewise Polynomial Collocation Method for the Solution of Fredholm Equation of Second Kind by Using AGE Iteration," *International Conference of World Engineering, Science and Technology Congress*, vol. 1123, p. 012039, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[8] E.S. Shoukralla, and M.A. Markos, "The Economized Monic CHEBYSHEV Polynomials for Solving Weakly Singular Fredholm Integral Equations of the First Kind," *Asian-European Journal of Mathematics*, vol. 13, no. 1, p. 2050030, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[9] C. Ravichandrana, K. Logeswari, and F.Jarad, "New Results on Existence in the Framework of Atangana–Baleanu Derivative for Fractional Integro-differential Equations," *Chaos, Solitons and Fractals*, vol. 125, pp. 194-200, 2019. [CrossRef] [Google Scholar] [Publisher Link]

[10] D. Codony et al., "An Immersed Boundary Hierarchical B-spline Method for Flexoelectricity," *Computer Methods in Applied Mechanics and Engineering*, vol. 354, pp. 750-782, 2019. [CrossRef] [Google Scholar] [Publisher Link]

[11] N.S. Mohamad, and J. Sulaiman, "The Piecewise Collocation of Second Kind Fredholm Integral Equations by Using Quarter-Sweep Iteration," *Journal of Physics: Conference Series*, vol. 1358, p. 012052, 2019. [CrossRef] [Google Scholar] [Publisher Link]

[12] Gregor J. Gassner, and Andrew R. Winters, "A Novel Robust Strategy for Discontinuous Galerkin Method in Computational Fluid Mechanicss: Why? When? What? Where?," *Frontier in Physics*, vol. 8, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[13] Shiva Sharma, Rajesh K. Pandey, and Kamlesh Kumar, "Collocation Method with Convergence for Generalized Fractional Integro-differential Equations," *Journal of Computational and Applied Mathematics*, vol. 342, pp. 419-430, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[14] Xiaoping Zhanga, Jiming Wub, and Dehao Yua, "The Superconvergence of Composite Trapezoidal Rule for Hadamard Finite-Part Integral on a Circle and Its Application," *International Journal of Computer Mathematics*, vol. 87, no. 4, pp. 855–876, 2010. [CrossRef] [Google Scholar] [Publisher Link]

[15] Klaus-Ju¨rgen Bathe, and Mirza M. Irfan Baig, "On a Composite Implicit Time Integration Procedure for Nonlinear Dynamics," *Computers and Structures*, vol. 83, pp. 2513–2524, 2005. [CrossRef] [Google Scholar] [Publisher Link]

[16] S. Karunanithi et al., "A Study on Comparison of Jacobi, Gauss-Seidel and Sor Methods for the Solution in System of Linear Equations," *International Journal of Mathematics Trends and Technology (IJMTT)*, vol. 56, no. 4, pp. 214-222, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[17] N.S. Mohamad, and J. Sulaiman, "The Piecewise Polynomial Collocation Method for the Solution of Fredholm Equation of Second Kind by SOR Iteration," *AIP Conference Proceedings*, vol. 2013, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[18] N.F.A. Zainal, J. Sulaiman, and M.U. Alibubin, "Application of EGSOR Iteration with Nonlocal Arithmetic Discretization Scheme for Solving Burger's Equation," *Journal of Physics: Conference Series*, vol. 1123, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[19] Yifen Ke, and Changfeng Ma, "On SOR-Like Iteration Methods for Solving Weakly Nonlinear Systems," *Optimization Methods and Software*, vol. 37 no. 1, pp. 320-337, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[20] Thai Son Hoang et al., "A Composition Mechanism for Refinement-Based Method," *2017 International Conference on Engineering of Complex Computer System*, 2017. [CrossRef] [Google Scholar] [Publisher Link]

[21] V.B. Kumar Vatti, Shouri Dominic, and S. Sahanica, "A Refinement of Successive Over Relaxation(RSOR) Method for Solving of Linear System of Equations," International *Journal of Advanced Information Science and Technology*, vol. 40, no. 40, pp. 1-4, 2015. [Google Scholar]

[22] G.D. Smith, *Numerical Solution of Partial Differential Equations: Finite Difference Method*, Clarendon Press: Oxford, 1985. [Google Scholar] [Publisher Link]

[23] Jia Kang, and Su Jingchun, "Analysis on Ternary Paradox of Fiscal Distribution: Theory and Mitigation Methods," *SSRG International Journal of Economics and Management Studies*, vol. 6, no. 11, pp. 63-72, 2019. [CrossRef] [Google Scholar] [Publisher Link]

[24] H.S. Ramane et al., "Numerical Solution of Fredholm Integral Equations Using Hosoya Polynomial of Path Graphs," *American Journal of Numerical Analysis*, vol. 5, no. 1, pp. 11-15, 2017. [CrossRef] [Google Scholar] [Publisher Link]

[25] Muhammad Mujtaba Shaikh, "Analysis of Polynomial Collocation and Uniformly Spaces Quadrature Methods for Second Kind Linear Fredholm Integral Equations- A Comparison," Turkish *Journal of Analysis and Number Theory*, vol. 7, no. 4, pp. 91-97, 2019. [CrossRef] [Google Scholar] [Publisher Link]

[26] N. Paradin, and Sh. Gholamtabar, "A Numerical Solution of The Linear Fredholm Integral Equations of the Second Kind," *Journal of Mathematical Extension*, vol. 5, no. 1, pp. 31-39, 2010. [Google Scholar] [Publisher Link]

[27] Federico Izzo, "*High Order Trapezoidal Rule-based Quadratures for Boundary Integral Methods on Non-parametrized Surfaces*," KTH Royal Institute of Technology, 2022. [Google Scholar] [Publisher Link]

[28] Sh. A. Meligy, and I.K. Youssef, "Relaxation Parameters and Composite Refinement Techniques," *Results in Applied Mathematics*, vol. 15, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[29] Sachine Bhalekar, and Varsha Daftardar-Gejji, "Convergence of the New Iterative Method," *International Journal of Differential Equations*, 2011. [CrossRef] [Google Scholar] [Publisher Link]

[30] N.A.M. Ali et al., "The Similarity Finite Difference Solutions for Two-Dimensional Parabolic partial Differential Equations via SOR Iteration," *Lecture Notes in Electrical Engineering*, vol. 724, pp. 515-526, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[31] L.H. Ali, J. Sulaiman, and S.R.M. Hashim, "Numerical Solution of SOR Iterative Method for Fuzzy Fredholm Integral Equation of Second Kind," *American Institute of Physics*, vol. 2013, no. 1, 2018. [CrossRef] [Google Scholar] [Publisher Link]