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Original Article

Vibration Analysis of an Un-Cracked and Cracked Euler-Bernoulli Simply Supported Beam

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Abstract - *The existence of cracks in a structural member such as a beam results in a change in its physical characteristics, which inaugurates flexibility and thus decreases the stiffness of the structural member with an intrinsic reduction of mode shape natural frequencies. In consequence it leads to alteration in the dynamic response of the beam. This paper focuses on the theoretical investigation of the lateral vibration of an uncracked, simply supported beam and five-mode shape frequencies are explored. The uncracked, simply supported beam is customized by using the Euler-Bernoulli beam theory. Numerical results obtained through Finite Element Analysis software – "Ansys Workbench 17.0", are used to compare with the theoretical values, and the percentage error between the two values is determined. Additionally, a model of the cracked simply supported beam with an open edge crack has been presented and free vibration analysis is done. The study investigates how mode shapes natural frequencies are altered due to the presence of cracks at different locations and with changeable depths.*

Keywords - *Simply supported beam, Free vibration, Mode shape natural frequency, Crack, FEM.*

1. Introduction

As a result of operating under loading conditions, most of the engineering structural members may be subjected to damages or cracks in overstressed areas. The formation of a crack indicates the beginning of failure in engineering structures; hence, it is very important to detect cracks in a structure as soon as they appear. When there are cracks in a beam or other structural parts, the location and depth of the cracks are mostly responsible for local variation in stiffness. The physical properties of a structure, like a beam, are altered by the existence of cracks that change the dynamic response characteristics of a structure. A structure's performance, safety and integrity can be assessed by monitoring the changes in the response parameters. Irregularities in vibration response characteristics can be observed based on the state of the crack that is open, closed or breathing in nature. For a long time, many researchers have investigated the vibration behavior of cracked structures, and they have been working hard to find methods which are cheaper and more reliable that can detect damage and monitor structural health, especially for the detection of cracks. One of these proper methods is finding cracks by analyzing the vibration behavior of structures having the existence of cracks. Most of the published papers assume that during vibration, cracks always remain open in a structural member. When dynamic loadings are dominant, this assumption may seem to be invalid. In such situations, the crack opens and closes regularly, which leads to variation in structural stiffness and nonlinear dynamic behavior will exist.

The proper performance of a machine is seriously threatened due to the presence of cracks. Material fatigue is the prime cause of most of the equipment's failures. Hence techniques that can early identify and localize the cracks have been the focus of several studies.

2. Literature Review

Research has been conducted at various research institutes across the world [1-9]. Some localized differences in stiffness in a structural element result in fractures that have some significant impact on the dynamics of the entire system. The existence of cracks leads to changes in the frequencies of natural vibrations, amplitudes of forced vibrations and dynamic stability [1-9]. It is possible to find the cracks without dismantling the entire system by analyzing the changes due to the presence of cracks. In a distinguishable work, Dimarogonas [10] reviewed several crack modeling approaches like equivalent reduced cross-section, local flexibility and local bending moment, as well as crack identification methods in beams and rotors using analytical, numerical and experimental methods. Christides and Barr [11] obtained the differential equation supported by related boundary conditions for a beam having one or more symmetric cracks by using the Euler-Bernoulli beam theory. A function was used to introduce the uncertainty in stress caused by the presence of a crack. This function contains a parameter which was evaluated by means of experimental tests and decayed exponentially with the distance from the crack location. They showed how closely the outcomes of their theoretical investigation matched with experimental value. They considered a series of comparison functions that consist of mode shapes of the corresponding undamaged beam. Their approach enables one to determine the higher natural frequencies and mode shapes of the cracked beam.

They used the two-dimensional finite element method to validate their theoretical approach. Shen and Pierre [12] extended the research work done by Christides and Barr [11] by incorporating an approximate Galerkin solution in the case of beams having pairs of symmetric open cracks. A set of comparison functions comprising the mode shapes of the similar undamaged beam were taken into consideration. By using their method, anyone can find out the higher natural frequencies and mode shapes of a crack beam. To validate their theoretical approach, they took the help of the 2D finite element method.

Chondros et al. [13] established the continuous beam vibration theory of having single-edge or double-edge open cracks using the Euler– Bernoulli beam theory. They considered the cracked beam as 1D continuum media and obtained the differential equation and the boundary conditions by using the Hu–Washizu–Barr variational formulation. By using the fracture mechanics method, the crack was designed as a continuous flexibility having a displacement field around the crack. They used two cases: (1) an aluminum beam having cracks due to fatigue and (2) a beam which is made of steel with a double-edge crack. They derived first natural frequency of the cracked beam and authenticated their findings with experimental results. Ostachowicz and Krawczuk [14] evaluated the natural frequencies of a cantilever beam having two open cracks. They considered that single-sided cracks are subjected to variable loading and double-sided cracks are subjected to cyclic loading. All the investigations are based on the assumption that cracks will remain open during loading.

Masoud et al. [15] considered a prestressed beam having fixed–fixed boundary conditions and investigated the effect of crack depth on the transverse vibration of the beam. With the help of modal analysis, they concluded that the axial load and the crack depth have a significant coupling effect, which affects the natural frequency of the beam. Their theoretical analysis was validated by experimental investigation. Shifrin and Ruotol [16] proposed a novel technique that can determine a beam's natural frequencies having an arbitrary finite number of open cracks.

With the help of a new technique, they were able to decrease the dimension of the calculation matrix, which helped to reduce the computation time compared to standard techniques for the continuous beam model. Kisa and Gurel [17] introduced a novel numerical approach in order to analyze the free vibration analysis of a cracked beam having uniform and stepped circular cross-sections, respectively. Their analysis was performed by using finite element and component mode synthesis methods together. Their assumption was that from crack locations, the beam is detached into segments, and these segments are connected by incorporating the flexibility matrices, which are derived from fracture mechanics. Zheng and Kessissoglou [18] used the finite element method in order to determine the natural frequencies and mode shapes of a cracked beam. Their approach provided more precise mode shapes because they established a shape function that could meet the local flexibility requirements at the crack positions. Fernandez-Sa'ez et al. [19] developed a simpler approach for evaluating the closed form of fundamental frequency and mode shape of Euler-Bernoulli beams with a single crack by using Rayleigh's method. The effect of the crack on the function of the undamaged beam was represented by incorporating a polynomial function. Although their method attracted good accuracy for the first natural frequency, they were unable to address the accuracy of their method for determining the first mode shape.Zhong and Oyadiji [20] expanded the method developed by Fernandez-Sa'ez et al. [19] in order to calculate both mode shapes and natural frequencies of a simply supported beam with a stationary roving mass and a crack. Lin and Chang [21] discovered the natural frequencies and mode shapes of a cantilever beam having a single crack by using the transfer matrix method.Khorram et al. (2012) [22] compared the performances of two damage detection approaches, which are wavelet-based, in order to find the size and location of a simply supported beam that is subjected to moving load.

Khorram et al., 2012 [23] developed a new method which enables to detection of multiple cracks in a simply supported beam when a moving load is acting along the beam length by using Continuous Wavelet Transform in combination with factorial design methods. They considered the beam deflection only when the moving load is passing the mid-length of the beam.The above studies focus on cracked beams having either single or double edge cracks. In some literatures, the vibration characteristics are measured when a single edge crack is varying in location or depth. Some literatures show how beam vibration characteristics are changing due to two edge cracks' variation in depth. The nature of variation of vibration characteristics is different based on both location as well as thickness changing effects as the natural frequency of vibration is dependent on the stiffness of the beam under lateral loading; both location and depth variation cause the effect of changing the beam stiffness. In most of the reported studies on cracked beams, the variation of natural frequency with both crack location and crack depth is not emphasized. Hence, there is still a research gap on vibration characteristics when both crack depth and location are changing. The present study considers the effect of both crack location and depth on the natural frequency on simply supported beams up to the fifth mode. Here Euler Bernoulli beam theory is used to find natural frequency as the proposed beam is thin beam.

3. Method

3.1.Theoretical Analysis of Transverse Vibration of Simply Supported Beam

A simply supported beam is said to that one where one end is hinged, and the other end is roller-supported. Elementary theory of bending of beams, also known as Euler-Bernoulli beam theory, the bending moment and transverse deflection can be expressed as:

$$M=EI\frac{d^2y}{dx^2}$$

For uniform beam equation of motion can be expressed as:

$$\frac{EI}{\rho A}\frac{d^4y}{dx^4}+\frac{d^2y}{dt^2}=0 \quad (1)$$

$$c^2\frac{d^4y}{dx^4}+\frac{d^2y}{dt^2}=0 \ (c=\sqrt{\frac{EI}{\rho A}}) \quad (2)$$

The solution of Eq. (2) is depended on both position and time.

So, the solution will be $y = w(x)T(t)$ (3)

Hence, Eq. (2) becomes:

$$\frac{c^2}{w(x)}\frac{d^4w(x)}{dx^4}=-\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} \quad (4)$$

$$\frac{d^4w(x)}{dx^4}-\beta^4w(x)=0 \quad (5a)$$

$$\frac{d^2T(t)}{dt^2}+\omega_i{}^2T(t)=0 \ (\beta^4=\frac{\omega_i{}^2}{c^2}=\frac{\rho A\omega_i{}^2}{EI}) \quad (5b)$$

To solve the above differential equations(5a,5b),$w(x)$ can be considered as: $w(x) = C_1\cosh(\beta x)$

$$+C_2\sinh(\beta x)+C_3\cos(\beta x)+C_4\sin(\beta x) \quad (6)$$

In order to solve Eq. (6), four boundary Equations are necessary:

For both support ends: $w(0, L) = 0$, $\frac{d^2y}{dx^2}(0, L) = 0$

Using Eq. (6) representing mode shapes and boundary conditions, the following relationships are derived:

$$C_1+C_3=0 \quad (7)$$

$$C_1\cosh(\beta L)+C_2\sinh(\beta L)+C_3\cos(\beta L)+C_4\sin(\beta L)=0 \quad (8)$$

$$C_1-C_3=0 \quad (9)$$

$$C_1\cosh(\beta L)+C_2\sinh(\beta L)-C_3\cos(\beta L)-C_4\sin(\beta L)=0 \quad (10)$$

Finally, from equations (7,8,9,10), the expression will be $\sinh(\beta L)\sin(\beta L) = 0$. This transcendental equation gives an infinite no. of natural frequencies of transverse vibration.

$$(\omega_i): \omega_i = (\beta_i L)^2\sqrt{\frac{EI}{\rho AL^4}}$$

The first five roots of the Eq. (6) are shown in Table 1. The dimensions and the material constants for the uniform simply supported beam investigated in this paper are as per ref. [24]: E=28 GPa, L=10 m, b=0.2 m, h=0.6 m, I=3.6× 10^{-3} m^4, m=282 kg, ρ= 2350 kg/m^3, ν=0.3.

The required five natural frequencies obtained are shown in Table 2. FEM (Ansys Workbench 17.0) is used to find the numerical result of mode shape natural frequencies that are shown in Table 3.

The error percentage between the two results is shown in Table 4.

Table 1. Value of roots

Roots	βiL
1	3.1416
2	6.2832
3	9.4248
4	12.5664
5	15.7080

Table 2. Mode shape natural frequency (Hz)

Mode	Frequency (Hz)
1	9.3913
2	37.5652
3	84.5217
4	150.2608
5	234.7825

Table 3. Mode shape frequency

Mode	Frequency (Hz)
1	9.2424
2	36.9200
3	83.6750
4	149.2508
5	232.2300

Table 4. Percentage error (%)

Mode	Theoretical Frequency (Hz)	Numerical Frequency (Hz)	Percentage Error (%)
1	9.3913	9.2424	0.947
2	37.5652	36.9200	1.71
3	84.5217	83.6750	1.00
4	150.2608	149.2508	0.74
5	234.7825	232.2300	1.08

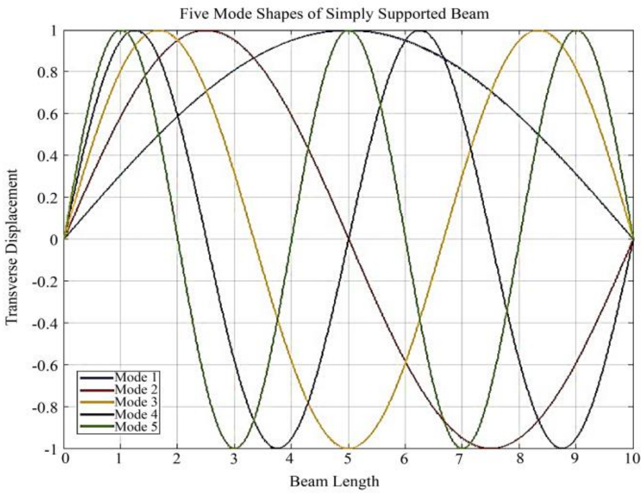

Fig. 1 Five mode shapes of simply supported beam

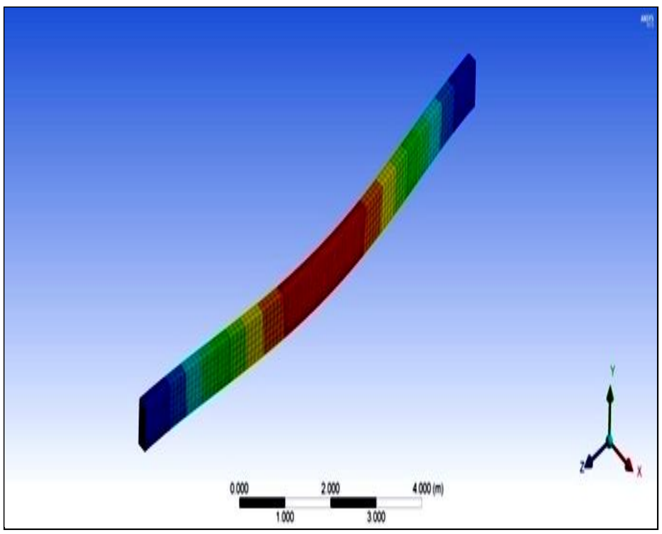

Fig. 2 First mode shape of simply supported beam

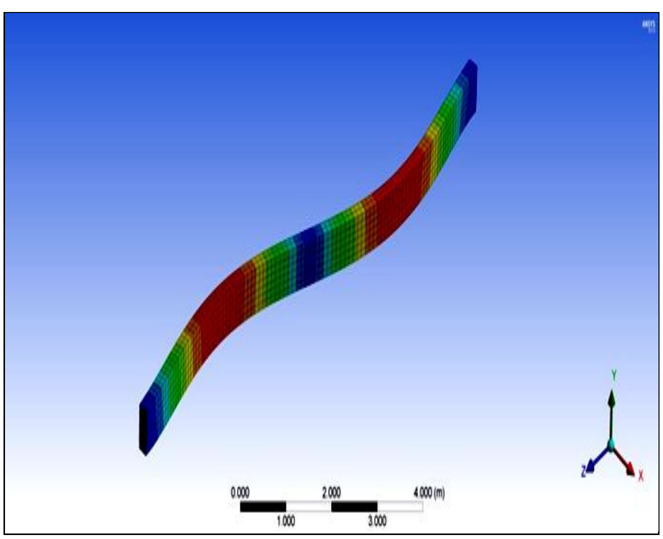

Fig. 3 Second mode shape of simply supported beam

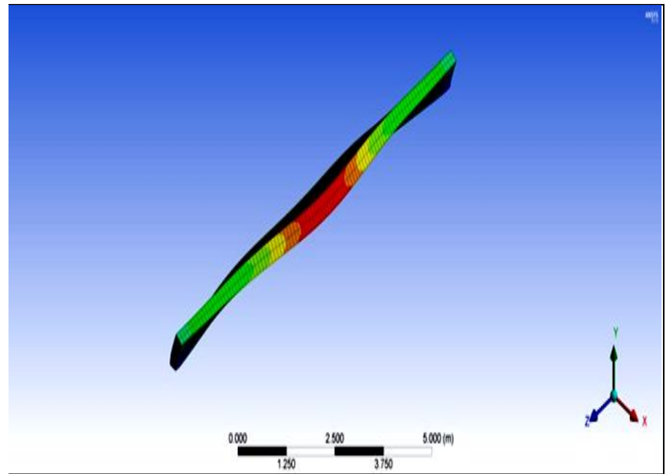

Fig. 4 Third mode shape of simply supported beam

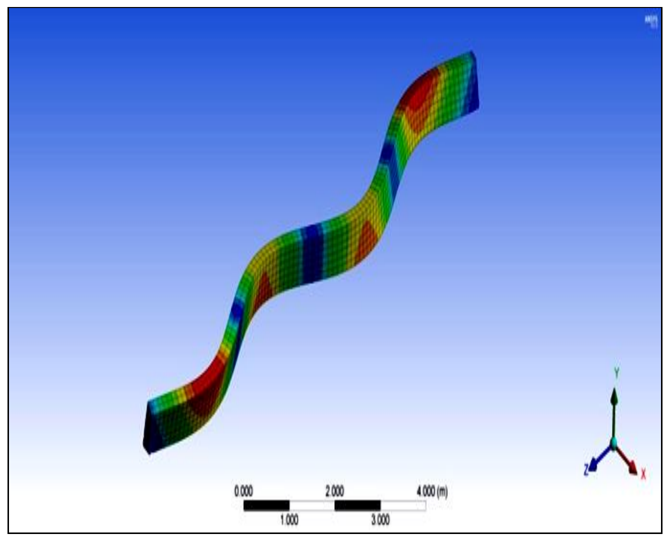

Fig. 5 Fourth mode shape of simply supported beam

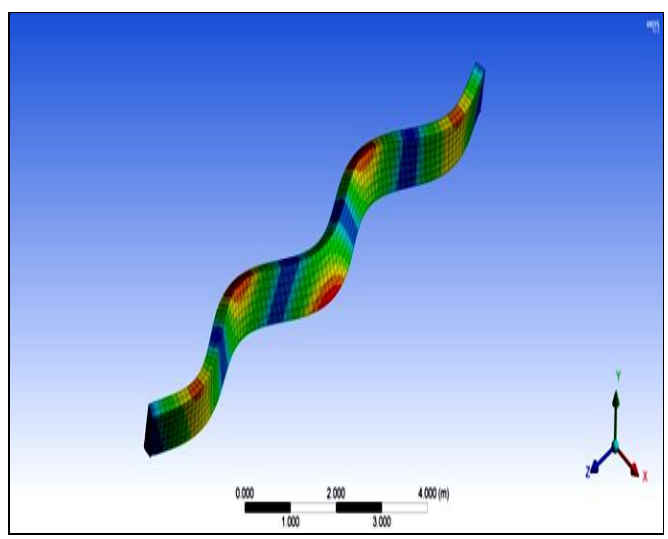

Fig. 6 Fifth mode shape of simply supported beam

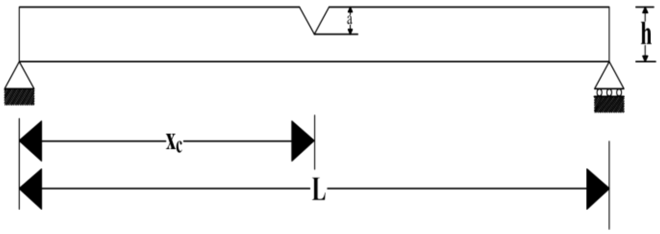

Fig. 7 Simply supported beam with a crack on the edge

3.2. Crack Modelling

The geometrical properties of structures like beams change due to crack formation and the study of its effect also becomes very complex. The modeling of crack has been a very important aspect. Finite Element Method is widely used for the analysis of mode shape natural frequency vibration. FEM software ANSYS Workbench 17.0 is used.

A simply supported cracked beam has been modeled, and freevibration analysis has been done by considering geometric and material linearity. The crack is considered to be an open-edge notch. A crack with a fixed width of 0.5 mm on the top surface of the beam has been modeled. It is assumed that uniform crack depth is present across the whole width of the beam.

4. Results and Discussions

The variations of the natural frequency of the cracked beam with varying crack depth for different crack locations and mode shapes are analyzed:

From the below graph it is observed that the drop in natural frequency is maximum when the crack is present at the middle position of the beam. When the crack is located at a position of 0.2 and 0.3 times of length from the left hinged support, the drop in natural frequency is minimal.

From Figure 8, it is clear that the amplitude of vibration is maximum at the mid-span of the beam. Hence natural frequency drop will also be maximum when the crack is present at that location as compared to the other two locations of 0.2L and 0.3L.

Table 5. Natural frequency variation for the first mode: ω=9.3024 Hz

Crack Position ($Ç_c = x_c/L$)	Crack Depth Ratio (H=a/h)	Natural Frequency Ratio (ω_c/ω)
Un-cracked beam		1.0000
0.2	0.1	0.9979
	0.3	0.986
	0.5	0.940
0.3	0.1	0.9976
	0.3	0.9633
	0.5	0.8878
0.5	0.1	0.8704
	0.3	0.8407
	0.5	0.8074

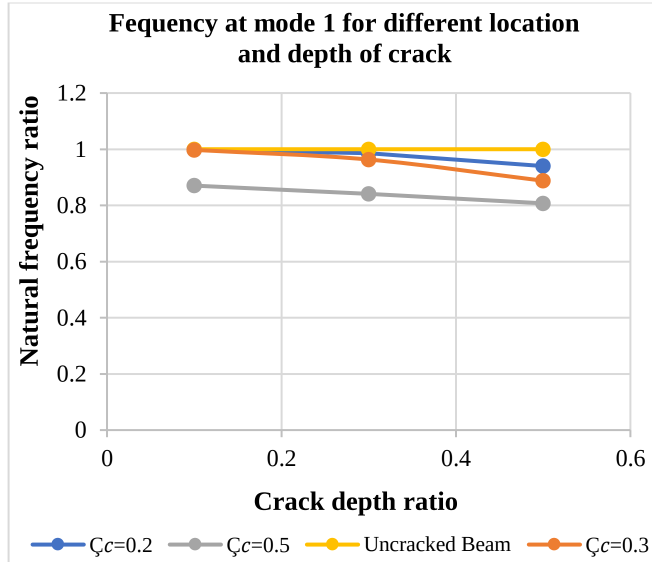

Fig. 8 Natural frequency ratio vs crack depth ratio for the first mode

Table 6. Natural frequency variation for the second mode: ω = 36.920 Hz

Crack Position ($Ç_c = x_c/L$)	Crack Depth Ratio (H=a/h)	Natural Frequency Ratio (ω_c/ω)
Un-cracked beam		1.000
0.2	0.1	0.9723
	0.3	0.9633
	0.5	0.9465
0.3	0.1	0.9701
	0.3	0.9632
	0.5	0.9485
0.5	0.1	0.9757
	0.3	0.9755
	0.5	0.9729

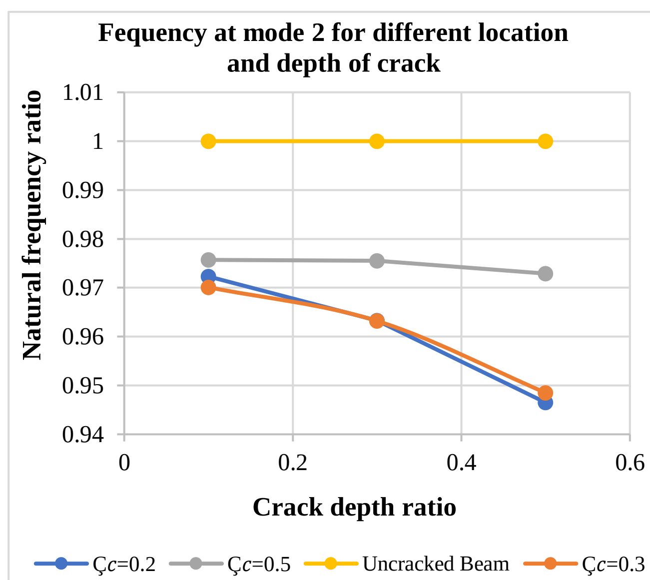

Fig. 9 Natural frequency ratio vs crack depth ratio for the second mode

From the above graph, it is seen that the drop in natural frequency is maximum when the crack is present at the position of 0.2 and 0.3 times of length from the left-hinged support of the beam. When the crack is located at the middle position of the beam, the drop in natural frequency is minimal.

From Figure 9, it is clear that the amplitude of vibration is maximum at a position of approx. 0.2 and 0.3 times of length from the left hinged support of the beam. Hence, the natural frequency drop will also be maximum at that position.

From the graph as shown below it is observed that the drop in natural frequency is maximum when the crack is present at the mid position of the beam. When the crack is located at a position of 0.2 and 0.3 times of length from the left hinged support, the drop in natural frequency is less as compared to mid position of the length.

From Figure 10, it is clear that the amplitude of vibration is maximum at the middle span of the beam. Hence natural frequency drop will also be maximum when the crack is present at that location as compared to the other two locations of 0.2L and 0.3L.

Table 7. Natural frequency variation for the third mode: ω=83.675 Hz

Crack Position ($Ç_c=x_c/L$)	Crack Depth Ratio (H=a/h)	Natural Frequency Ratio (ω_c/ω)
Un-cracked beam		1.000
0.2	0.1	0.9820
	0.3	0.980
	0.5	0.9761
0.3	0.1	0.9926
	0.3	0.9817
	0.5	0.9791
0.5	0.1	0.9745
	0.3	0.9684
	0.5	0.9648

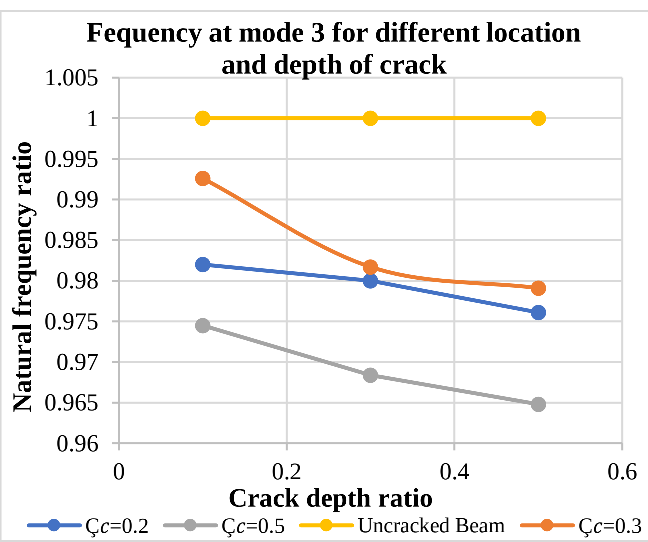

Fig. 10 Natural frequency ratio vs crack depth ratio for the third mode

For the fourth mode, it is clear that the drop in natural frequency is approximately the same for all the locations of the crack because, in all those positions, the amplitudes of vibration are maximum, as shown in Figure 11.

Table 8. Natural frequency variation for the fourth mode: ω=149.2508 Hz

Crack Position ($Ç_c=x_c/L$)	Crack Depth Ratio (H=a/h)	Natural Frequency Ratio (ω_c/ω)
Un-cracked beam		1.000
0.2	0.1	0.9926
	0.3	0.9910
	0.5	0.9897
0.3	0.1	0.9924
	0.3	0.9917
	0.5	0.9902
0.5	0.1	0.9922
	0.3	0.9906
	0.5	0.9886

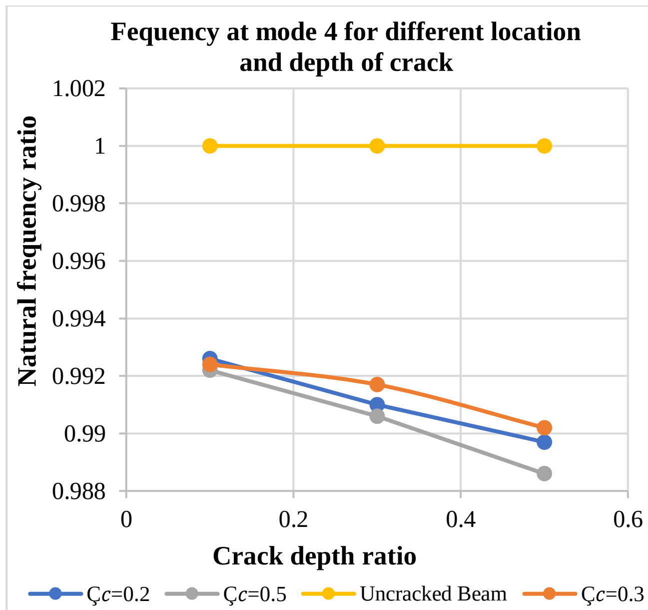

Fig. 11 Natural frequency ratio vs crack depth ratio for the fourth mode

Table 9. Natural frequency variation for the fifth mode: ω=232.23 Hz

Crack Position $Ç_c=(x_c/L)$	Crack Depth Ratio H=(a/h)	Natural Frequency Ratio (ω_c/ω)
Un-cracked beam		1.000
0.2	0.1	0.9952
	0.3	0.9913
	0.5	0.9892
0.3	0.1	0.9932
	0.3	0.9901
	0.5	0.9858
0.5	0.1	0.9877
	0.3	0.9871
	0.5	0.9853

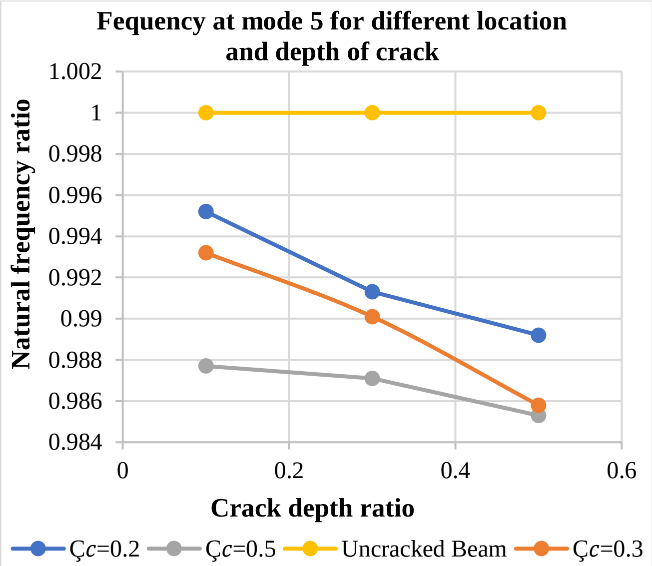

Fig. 12 Natural frequency ratio vs crack depth ratio for the fifth mode

For the fifth mode of vibration, natural frequency drop is maximum for mid length of the crack location.

The other two locations of crack show less drop in natural frequency because the amplitude of vibration is less as compared to the middle position of length, which is evident in Figure 12.

5. Conclusion

- Substantial changes in natural frequency are observed based on the location of cracks as well as the size of cracks.
- Natural frequencies of cracked simply supported beams at a particular location are inversely proportional to the depth of the crack when crack positions are constant.
- It has been noticed that the change in frequencies is a function of both crack depth and crack location as well as of the mode number.
- The largest effects are noticed at the center of the beam, which indicates that when the bending moment is higher, a decrease in frequencies is predominant at that position.

Nomenclature

M	Bending moment
ν	Poisson's ratio
L	Beam length
B	Beam width
H	Beam height
t	time
ρ	Mass density of the beam
y (x, t)	Deflection of the beam
x	Location along the beam length
E	Young modulus of elasticity of beam material
$\zeta_c = \frac{x_c}{L}$	Non-dimensional crack location ratio
I	Area moment of inertia of the beam c/s
$H = \frac{a}{h}$	Non-dimensional crack depth ratio
x_c	Crack location along the beam length
m	Mass of the beam per unit length
a	Crack depth from the edge of the beam
A	Cross-sectional area of the beam
ω_i	ith natural frequency of the un-cracked beam
ω_c	Natural frequency of the cracked beam

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