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Original Article

Transient Analysis of Circuits with Memory Elements Modelled using Fractional Variable-Order Dynamics: A Scarpi Derivative Approach

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Abstract *- This work attempts to extend classical and fractional-order circuit models to include time-varying memory effects through Variable-Order (VO) derivatives. Traditional circuit elements are governed by integer-order dynamics, while fractional-order models capture non-ideal behaviors using fixed fractional orders. The dynamics of memory element inductor is modelled using the Scarpi definition of the VO derivative. By adopting the Scarpi definition of VO derivatives, this study enables consistent modeling of circuits whose dynamic characteristics evolve over time. The analysis of homogeneous (natural response) and nonhomogeneous (forced response) of these circuits reveals that the choice of derivative function and its parameters directly influences the time evolution of the current in the circuit, especially the transient features like rise time and damping, as well as steady-state behavior. Simulation results demonstrate that VO-based models provide a more flexible and realistic framework for representing complex circuit dynamics compared to conventional and fractional-order approaches.*

Keywords *- Fractional-Order systems, Variable-Order calculus, VO modelling, Linear systems, Scarpi variable-order derivatives, Electric circuit, Time-Domain analysis.*

1. Introduction

Fractional calculus (calculus with integrals and derivatives with non-integer order) is now a quite matured field with applications ranging from biology, financial systems, thermodynamics, physics, medicine and many more [1]. The non-local properties of Fractional-Order (FO) derivative operators support them with an upper hand over the classical local Integer-Order (IO) derivatives. Many real-world and engineering systems have been modelled more realistically with the help of FO Differintegral Equations (FDEs). In domains such as control systems, machine learning, signal and image processing, network analysis, artificial neural networks also, Fractional Calculus (FC) is used as an application extension, and for more details, refer to [2-8].

In recent years, extensive research has been conducted, and FC has been expanded to include higher derivatives such as complex-order derivatives as well as conjugated complex-order derivatives. Extensive work is done for fractional-order linear systems as discussed in [9-11]. Analysis and modelling of electrical circuits using fractional elements is elaborated in [12, 13]. Recently, for enormous industrial applications, FO Proportional Integral Derivative (FOPID) controllers are more popular. The pioneering work of FOPID is reviewed in [14], and tuning methods and FOPID design are discussed in [16-22]. The Variable-Order (VO) derivatives are marked by peculiarity of extending the constant FO derivative order by a time-dependent function. As the literature suggests, this generalization of the derivative order to a time-varying function endows the resulting mathematical model with greater flexibility, accuracy, and compact representation of the underlying physical process [23]. Certain applications of fractional calculus require that the order of integration or differentiation is not kept constant and is variable with respect to a parameter of the system. variable-order derivative is the order of differentiation as a function/variable. It is achieved through using variable-order differential equations, which are mostly used in the case of problems of dynamically varying regimes like variable viscoelastic oscillators. In complex modelling/approaches to the modelling of anomalous diffusion the variable-order derivative model is applied as an important instrument [24, 25]. The variable-order derivatives and VO-FC derivatives have found application in different fields, such as mechanics where they are used in discussing the viscoelasticity, constitutive relations, development of viscoelastic oscillators, modelling of transport processes, anomalous diffusion issues, control processes, ecological models and biological ones, investigation of the random-order models, and many more. Various applications of VO

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derivative models have been reported in the literature [26, 27]. Most of the literature regarding applications of variable-order derivatives considers variable-order derivatives to be an extension of fractional-order derivatives, in which the constant FO derivative α has been replaced with a function which varies with time as $\alpha(t)$, where, $\alpha \in \mathbb{R}^+$ by the variable time function $\alpha(t)$: $[0, t] \rightarrow (0,1)$. However, as reported by [32], Variable-order integral and derivative operators derived using straightforward extensions lack a rigorous mathematical framework. In particular, the variable-order derivative operator and the variable-order integral operator defined by the direct extension which do not conform to the rules of the basic theorem of calculus, whereby the variable-order derivative operator should behave as the left inverse of the variable-order operator of integral. To address this issue, mathematically sound and an alternate definition of variable-order integrals and derivatives based on the Scarpi derivatives is discussed in [28]. The Scarpi Variable-Order Fractional (VOF) integrals and derivatives operators satisfy the fundamental theorem of calculus and are formulated based on the Laplace-domain representation of their corresponding Fractional-Order (FO) operators and Sonine conditions [23-34].

Based on this observation and the preceding literature survey, the application of Scarpi definition of VO operators in modelling real-world systems is still a largely unexplored area. The noticeable gaps that can be identified include exploiting the sound mathematical basis of Scarpi VO derivatives in modelling and in engineering applications, development of analytical solution methods for VOFDEs, characterization of the memory elements in electrical circuits using these new definitions of VO operators. This work attempts to analyze the behaviour of electrical circuit with memory elements modelled using VO derivatives in the Scarpi sense.

The electrical circuits with a series combination of Resistance (R) and a memory element (Inductor L or Capacitor C), commonly known as series RL or RC circuits, are fundamental representations of systems with one memory element. The analysis of these basic circuits provides interesting insights of the dynamics of such systems [15]. The notion of series RL/RC circuits has been extended to FO domain with L or C represented by a FD definition. The conventional RL and RC circuits in a fractional-order sense is generalised in [7]. A numerical approximation of the Caputo–Fabrizio fractional derivative for fractional orders ranging from value 1 to 2 is proposed. Furthermore, the conventional RLC circuit is modified by replacing the ordinary derivative with a fractional-order derivative [5].

The analysis of electrical circuits comprising conventional one-port elements, such as RC and RL networks, is discussed in [12]. In this paper, analytical relationships describing the voltage, current, and power characteristics of the proposed fractional-order elements are established, and their practical realization is achieved using ladder-type electrical circuits. Numerical and analytical results after considering different sources in fractional equation are presented in [13].

The results show that the modified capacitance, inductance, and resistance display time-dependent fluctuations with fractal behavior at different scales. Additionally, heterogeneities in the electrical components were identified, contributing to irreversible energy dissipation discussed in [17].

This novel work extends the results of [28] by analyzing different types of linear circuits, both homogeneous and non-homogeneous, using Scarpi VO derivative definitions. This approach also involves the numerical computation of Inverse Laplace Transform (ILT) of VO Transfer Functions (VOTFs). The main contributions of this work are given as:

a. Application of VO Scarpi derivatives for modelling memory element (inductor) in a circuit.
b. Transient analysis (natural response) of linear circuit with VO dynamics with non-zero initial conditions. Transient analysis (forced response) of linear circuit with VO dynamics with a source.
c. Study of effect of VO derivative function on the evolution of current conditions.

The paper layout is as follows. Section 2 brought attention to the basic definitions of Fractional Calculus (FC). In Section 3, variable-order Scarpi operators of differintegral are defined. In Section 4, results are showcased. The implications of these results are discussed extensively in Section 5. The result's conclusion is discussed in Section 6.

2. Fractional Calculus

Fractional Calculus (FC) is an extension of classical calculus that deals with integration and differentiation of arbitrary orders, including non-integer, real, and complex values. Owing to its ability to represent memory and hereditary characteristics, FC has gained considerable attention in the mathematical modeling of complex engineering systems and real-world phenomena. Over the past few decades, extensive theoretical investigations have been conducted on fractional-order differential equations [35-37], resulting in significant developments in this field. Consequently, fractional-order derivatives and integrals have found widespread applications across various disciplines of science and engineering. Some of the conventional definitions commonly used for fractional-order derivatives and integrals are presented below.

a. The fractional-order derivative using Grunwald-Letnikov (GL) derivative order definition for $\alpha \in \mathbb{R}^+$ as follows.

$$aD_t^{\alpha} f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \, {}^{\alpha}C_j \, f(t-jh), \quad (1)$$

Where the integer part of h is $[\frac{t-a}{h}]$ and the binomial coefficient is ${}^{\alpha}C_j$.

b. The fractional-order derivative using Riemann-Liouville (RL) derivative order definition $\alpha \in \mathbb{R}^+$ is given as

$$aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (2)$$

$\Gamma(\cdot)$ is the Gamma function and for $n > \alpha > n-1$, $with\ n \in \mathbb{Z}^+$.

c. The fractional-order derivative using Caputo definition for $\alpha \in \mathbb{R}^+$ is given as

$$aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (3)$$

for $n > \alpha > n-1$, where, $n \in \mathbb{Z}^+$, and n^{th}-order derivative is $f^n(\tau)$ of function $f(t)$.

Among researchers, the definition of a Caputo fractional derivative is mostly used and very much popular. The real-world systems representation in FO dynamics has been carried out by using a fractional derivative in Caputo sense and discussed in [38, 39].

In engineering systems, generally, Caputo fractional derivative is used to define signals and functions. Such Fractional-derivative equation model with Caputo fractional derivative allows initial conditions which are measurable and physical.

Various analytical and numerical methods which are easy to simulate and implementable are available to solve FDEs with Caputo fractional derivative, and such methods are given in [40]. The comparative study of Caputo and analysis of variable-order fractional derivative algorithms is given in [41, 44].

3. VO Scarpi Differintegral Operators

This section presents the definitions, which are discussed in [28] as fundamental definitions of variable-order Scarpi derivatives and integral operators. Consider an absolutely continuous function $f(t)$ defined over $[0,T]$. The Caputo fractional derivative can be represented as follows:

$${}^{C}D_0^{\alpha} f(t) = \int_0^t \phi(t-\tau) f'(\tau)\, d\tau, \quad (4)$$

Here, the weakly singular kernel is denoted by:

$$\phi(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad (5)$$

with $0 < \alpha < 1$,

Equation (4) defines the Caputo fractional derivative as a left-inverse operator for the Riemann-Liouville integral

$${}^{RL}I_0^{\alpha} f(t) = \int_0^t \psi(t-\tau)\, f(\tau)\, d\tau, \quad (6)$$

with the kernel

$$\psi(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \quad (7)$$

The definitions Equations (4) and (6) satisfies the basic calculus theorem,

$$CD_0^{\alpha\,RL} I_0^{\alpha} f(t) = f(t),$$

And

$${}^{RL}I_a^{c}\,{}^{c}D_a f(t) = f(t) - f(0).$$

For the kernels defined in Equations (5) and (7), the Laplace Transforms (LT) are given as

$$\phi(s) = \mathcal{L}\{\phi(t)\} = s^{\alpha-1}, \quad (8)$$

which can be rewritten as

$$\phi(s) = s^{s\mathrm{A}(s)-1},$$

Where

$$\mathrm{A}(s) = \mathcal{L}\{\alpha(t)\} = \mathcal{L}\{\alpha\} = \frac{\alpha}{s}, \quad (9)$$

Similarly,

$$\psi(s) = \mathcal{L}\psi(t) = \frac{1}{s^{\alpha}},$$

Which can also be represented as

$$\psi(s) = s^{-s\mathrm{A}(s)}, \quad (10)$$

With $A(s)$ as given in Equation (9), the Laplace transform of Caputo FO integral and derivative are used to establish equations Equations (4) and (6), respectively and can be rewritten as:

$$\mathcal{L}\{{}^{C}\mathrm{D}_0^{\alpha} f(t)\} = s^{s\mathrm{A}(s)} F(s) - s^{s\mathrm{A}(s)-1} f(0), \quad (11)$$

and,

$$\mathcal{L}\{{}^{RL}\mathrm{I}_0^{\alpha} f(t)\} = s^{-s\mathrm{A}(s)} F(s), \quad (12)$$

The aforementioned fractional-order operators can be extended to variable-order cases by representation in the Laplace domain. Consider $\alpha(t)$, as a function which is integrable locally,

$$\alpha(t): [0, T] \to (0,1),$$

Which has the Laplace transform,

$$\mathrm{A}(s) = \mathcal{L}(\alpha(t)). \quad (13)$$

Substituting Equation (13) into Equation (11) gives the Scarpi variable-order Caputo derivative, while substituting it into Equation (12) defines the corresponding Scarpi VO integral operator.

Thus,

$$\mathcal{L}\{{}^{S}\mathrm{D}_0^{\alpha(t)} f(t)\} = s^{s\mathrm{A}(s)} F(s) - s^{s\mathrm{A}(s)-1} f(0), \quad (14)$$

and,

$$\mathcal{L}\{{}^{S}\mathrm{I}_0^{\alpha(t)} f(t)\} = s^{-s\mathrm{A}(s)} F(s). \quad (15)$$

As shown in [28], the VO operators defined in Equations (14) and (15) satisfy the basic theorem of calculus, that is,

$${}^{S}D_0^{\alpha}(t)\, {}^{S}I_0^{\alpha}(t) f(t) = f(t)\,,$$

and,

$${}^{S}I^{\alpha(t)}\, {}^{S}D^{\alpha(t)} f(t) = f(t) - f(0).$$

These identities are valid provided that the kernels $\phi_\alpha(t)$ and $\psi_\alpha(t)$ constitute a Sonine pair, that is

$$\int_0^t \phi_\alpha(t-\tau)\, \psi_\alpha(\tau) d\tau = 1, \quad t > 0 \quad (16)$$

Which in Laplace domain is represented as

$$\phi_\alpha(s)\psi_\alpha(s) = \frac{1}{s}.$$

Further, as mentioned in [28], Equation (15) defines a commutative Scarpi variable-order integral operator that obeys the semigroup property.

4. Results

This work considers following two circuits:

a) Series RL circuit with initial energy in the inductor.

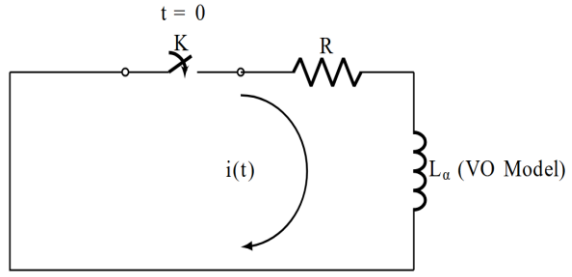

Fig. 1 Homogeneous VO series RL circuit

The inductor dynamics is modelled using VO Scarpi derivative (L_α) refer Figure 1. The inductor stores initial current i(0) = i_0 = 1Amp. The switch K is initially open and closes at t = 0. Also, for simplicity R = 1 ohm and L_α = 1 Volts/(Amp/S).

b) Series RL circuit excited by a voltage source v(t).

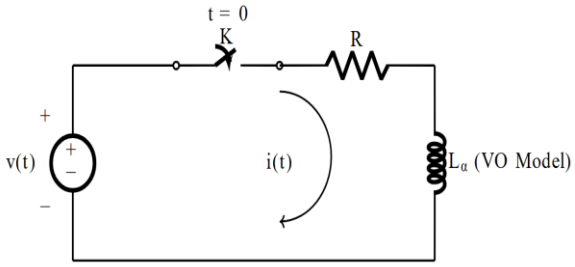

Fig. 2 Non-Homogeneous VO series RL circuit with source

Same values of R and L_α are considered with inductor in initially relaxed state (i_0 = 0). The switch K closes at t = 0 refer Figure 2. The remaining of the section presents a detailed analysis of the transient behavior of the aforementioned circuits under the influence of switching action and choice of variable derivative order $\alpha(t)$.

4.1. Analysis of Series RL Circuit without Source

Considering Figure 1, the circuit equilibrium condition yields following linear homogeneous Fractional VO Differential Equation (FVODE):

$$L_\alpha\, {}^{s}D_0^{\alpha} i(t) + R\, i(t) = 0, \quad (17)$$

With i_0 = 1. Laplace transform followed substitution of parameter values gives:

$$s^{s\mathrm{A}(s)} I(s) - i_0\, s^{s\,\mathrm{A}(s)-1} + R\, I(s) = 0,$$

Where, $A(s) = \mathcal{L}(\alpha(t))$.

This gives,

$$I(s) = \frac{i_0}{s(1 + R\, \psi_\alpha(s))}\,, \quad (18)$$

Where

$\psi_\alpha(s) = s^{-sA(s)}$.

In order to study the effect of VO function $\alpha(t)$ on the circuit response, this study considers five functions for $\alpha(t)$ as given in Table 1.

Table 1. Expressions for VO derivative order ($\alpha(t)$)

Derivative-order Case	α(t)
Case-1	$\alpha(t) = \beta\, e^{-at}$
Case-2	$\alpha(t) = \beta\,(1 + \sin t)$
Case-3	$\alpha(t) = \beta\, e^{-at}\,(1 + \sin t)$
Case-4	$\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)\, e^{-ct}$
Case-5	$\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)E_\beta(-ct^\beta)$

As seen, the choice of $\alpha(t)$ includes both exponentially decaying and sinusoidal functions.

4.1.1. Case-1

$\alpha(t) = \beta e^{-at}$, Where, $0 < \beta < 1$ and $a > 0$.

With the Laplace transform

$A(s) = \frac{\beta}{s+a}$, and $\psi_\alpha(s) = s^{\frac{-\beta s}{s+a}}$.

Substituting in Equation (18) gives,

$$I(s) = \frac{i_0}{s\left(1+R\, s^{\frac{-\beta s}{s+a}}\right)}. \quad (19)$$

As mentioned in [28], the analytical expression for the ILT of RHS of Equation (19) is not available. This leads to the choice of numerical ILT technique given in [42, 43]. In order to study the effect of $\alpha(t)$ on the circuit condition, following parameters are chosen: (see Table 2).

Table 2. Variations in parameters β and a

Sr. No	β	a
1	0.1	1
2	0.1	5
3	0.1	10
4	0.5	10
5	0.9	10

The time evolution of circuit current i(t) is shown in Figures 3 and 4.

Following observations are recorded:

a) The effect of β and a on the transient response is noted.
b) The choice of β and a values decides the speed of decay of i(t) with of β constant, increasing the value of a increases the decay rate of i(t).
c) With a constant, increase in the value of β alters the rate of decay in i(t). Interestingly, in this case, the value of β dominates the steady-state value at which i(t) settles.

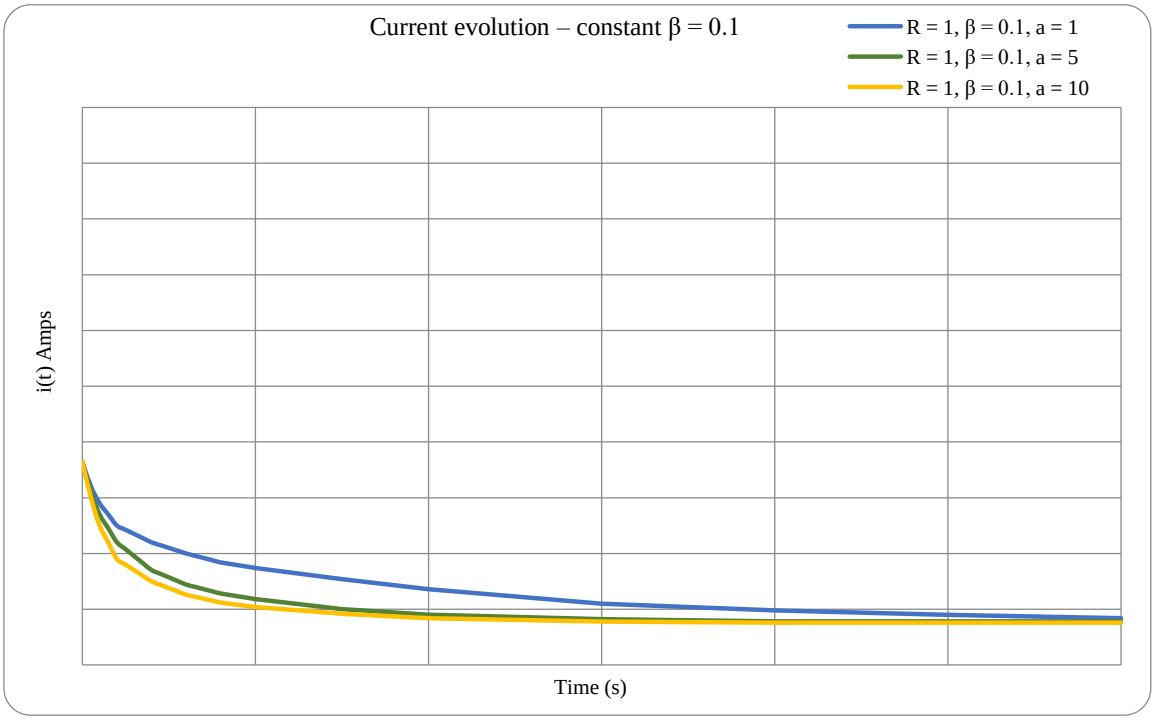

Fig. 3 Current evolution with respect to time considering initial condition as i_0 for homogeneous system variable-order FDE, $\alpha(t) = \beta e^{-at}$ with value β constant and vary value of a

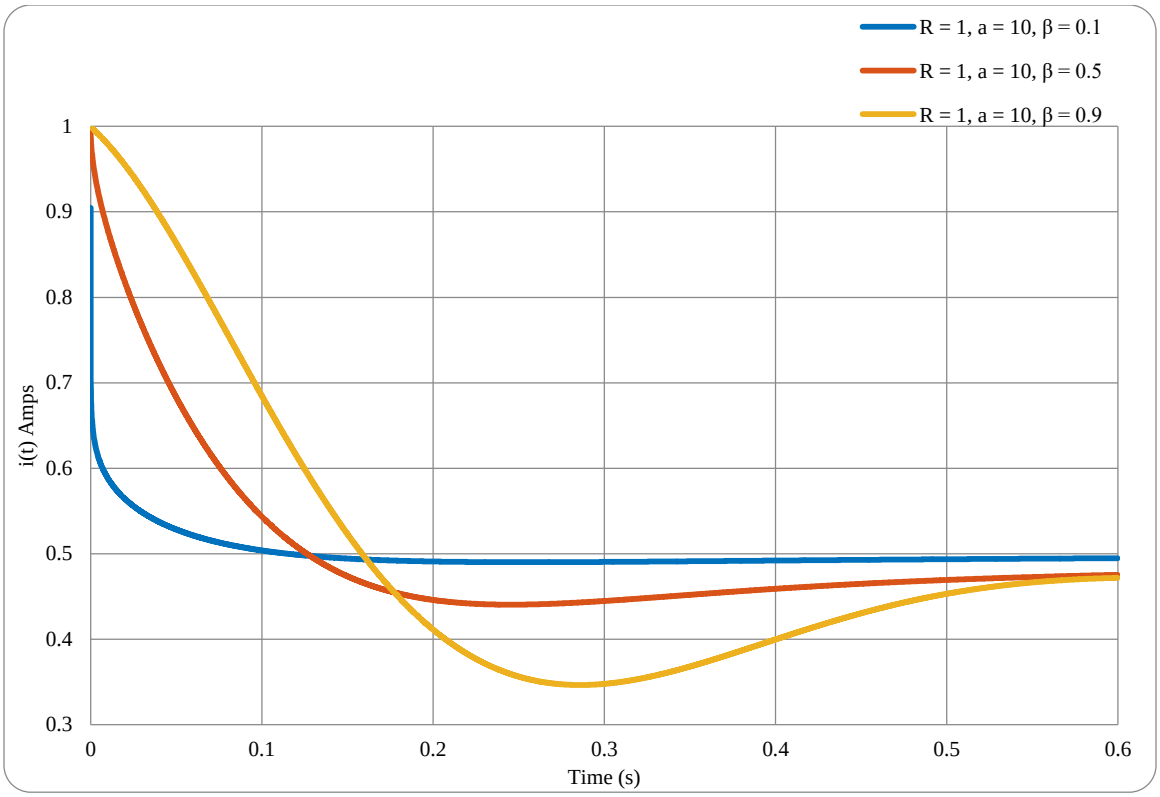

Fig. 4 Current evolution with respect to time considering initial condition as i_0 for homogeneous system variable-order FDE $\alpha(t) = \beta e^{-at}$ with value a constant and vary value of β

4.1.2. Case-2

$\alpha(t) = \beta(1 + \sin t)$,

Where, $\beta \in (0,0.5)$ with

$\psi_\alpha(s) = s^{-\beta\left[1+\frac{s}{s^2+1}\right]}$,

and $A(s) = \frac{\beta}{s+a}$.

Substituting the value of $\psi_\alpha(s)$ in Equation (18) gives,

$$I(s) = \frac{i_0}{s\left(1+R\, s^{-\beta\left[1+\frac{s}{s^2+1}\right]}\right)} \quad , \qquad (20)$$

And using the numerical ILT technique mentioned and used in [28], the current evolution with case-2 and the initial condition i_0 is obtained. In order to study the effect of $\alpha(t)$ on the circuit condition, the value of R is fixed as 1 and vary β values as shown in Table 3.

Table 3. Variations in parameters β

Sr. No	R	β
1	1	0.1
2	1	0.3
3	1	0.45
4	1	0.5

The time evolution of circuit current i(t) is shown in Figure 5.

Following observations are noted:

a) The effect of β on current evolution is noted from t = 0.

b) The decay rate of the current through the circuit i(t) depends upon the value of β. The value of i(t) settles to a steady-state value of 0.5Amp effectively by parameter β = 0.1.

c) Further, some transients are observed as sin t is a part of the function α(t).

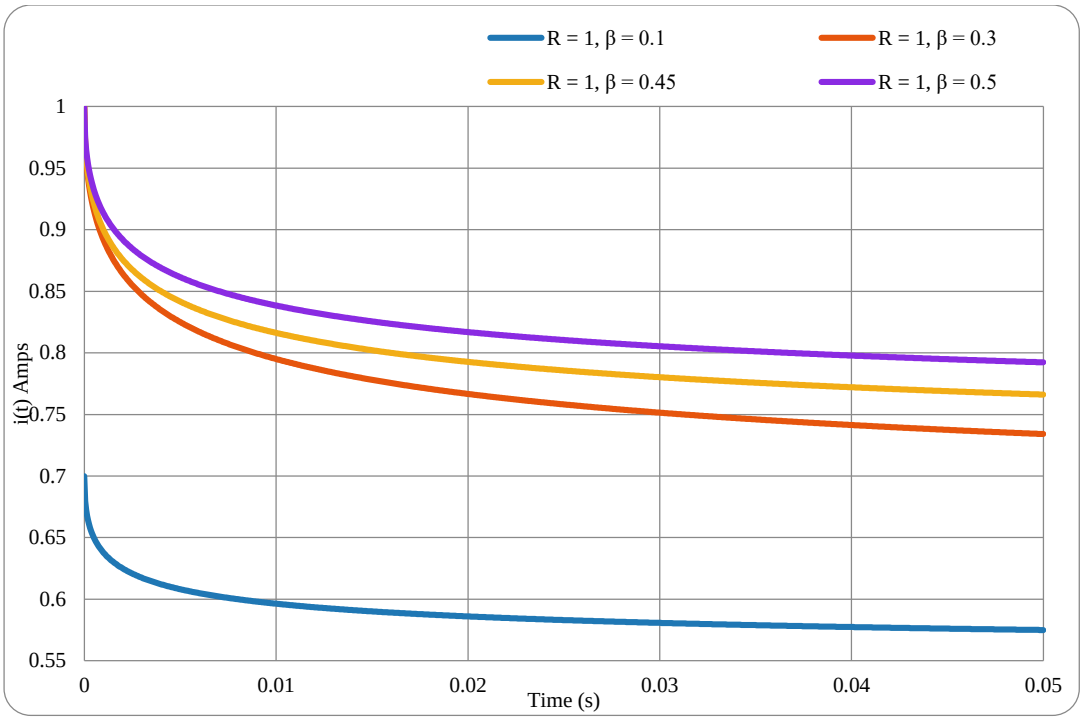

Fig. 5 Current evolution with respect to time considering initial condition as i_0 for homogeneous system variable-order FDE, $\alpha(t) = \beta(1 + \sin t)$

4.1.3. Case-3

$\alpha(t) = \beta e^{-at}(1 + \sin t)$,

Here $\beta \in (0,0.5)$ and $a > 0$.

With $\psi_\alpha(s) = s^{-s\beta[\frac{1}{s+a}+\frac{1}{(s+a)^2+1}]}$,

and $A(s) = \frac{\beta}{s+a}$.

Substituting in Equation (18) gives,

$$I(s) = \frac{i_0}{s\left(1+R\,s^{-s\beta[\frac{1}{s+a}+\frac{1}{(s+a)^2+1}]}\right)}. \quad (21)$$

To examine the influence of α(t) on the circuit behavior, the parameters listed in Table 4 are selected for further analysis.

Table 4. Variations in parameters β and a

Sr. No	β	a
1	0.1	1
2	0.1	5
3	0.1	10
4	0.3	10
5	0.45	10
6	0.5	10

The time evolution of circuit current i(t) is shown in Figures 6 and 7.

It is seen that:

a) The effect of β and a on the transient response are clearly visible.

b) With β being held constant, the value of a dominates the decay rate of the circuit current i(t).

c) With a constant, increasing the value of β, changes the decay rate of i(t). Notably, β also governs the steady-state value to which i(t) ultimately converges.

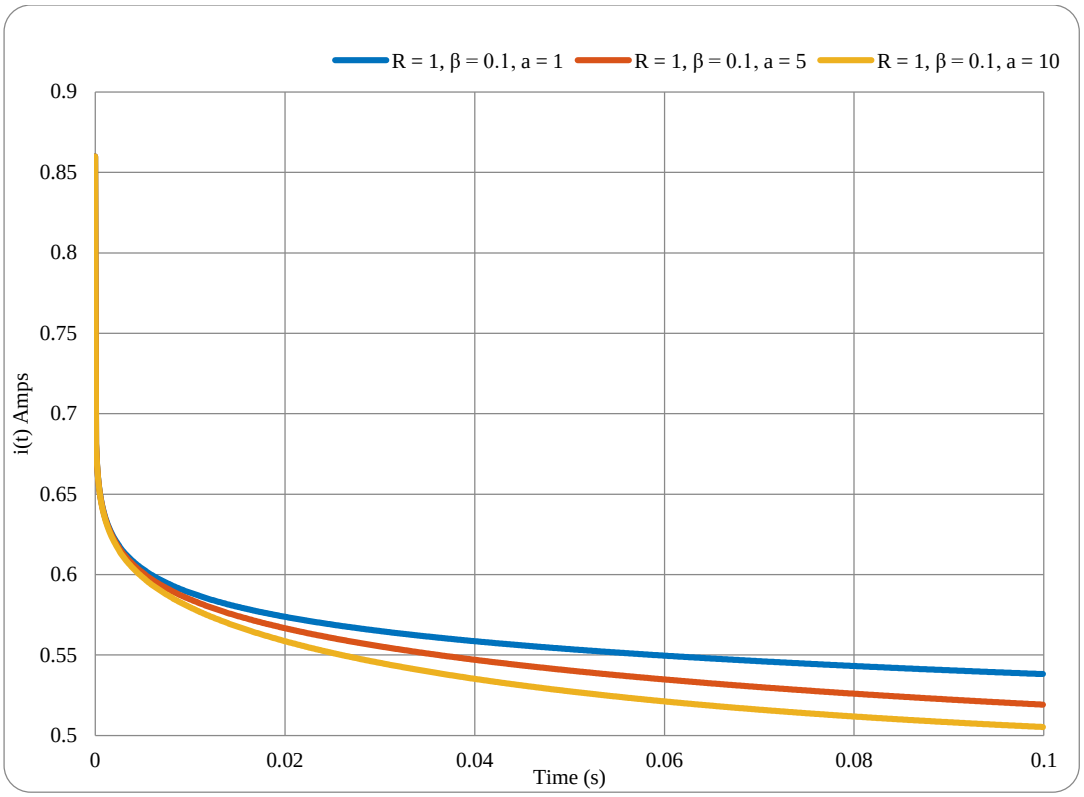

Fig. 6 Current evolution with respect to time considering initial condition as i_0 for homogeneous systems variable-order FDE, $\alpha(t) = \beta\, e^{-at}(1 + \sin t)$ consider constant β and vary value of a

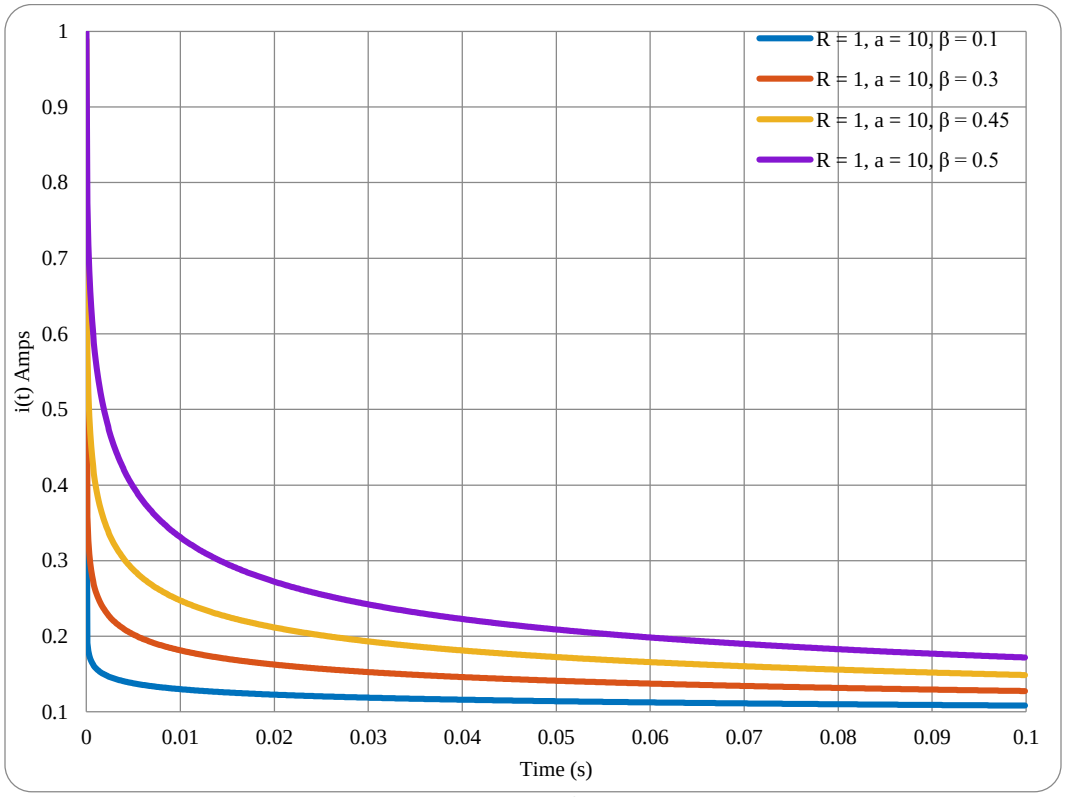

Fig. 7 Current evolution with respect to time considering initial condition as i_0 for homogeneous systems variable-order FDE, consider constant value a and vary value of β

4.1.4. Case-4

$\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)e^{-ct}$,

for $1 > \alpha_2 > \alpha_1 > 0$, and c > 0, a real constant,

Consider the function with

$\psi_\alpha(s) = s^{-\frac{\alpha_2 c + \alpha_1 s}{s(s+c)}}$,

and,

$A(s) = \frac{\alpha_2 c + \alpha_1 s}{s(s+c)}$.

Substituting in Equation (18) gives,

$$I(s) = \frac{i_0}{s\left(1 + R\, s^{-\frac{\alpha_2 c + \alpha_1 s}{s(s+c)}}\right)}, \qquad (22)$$

and using the numerical ILT technique, the transient response with case-4 and initial condition i_0 is obtained. To observe the influence of α(t) on the circuit behaviour, the parameters listed in Table 5 are selected for further analysis.

Table 5. Variations in parameters α_1, α_2 and c

Sr. No	α_1	α_2	c
1	0.6	0.8	0.5
2	0.6	0.8	2
3	0.6	0.8	3
4	0.4	0.6	3
5	0.5	0.7	3

The values of $R = 1$ is constant, and the response is shown for different values of α_1, α_2 and c in Figures 8 and 9.

The Following peculiarities in the behavior of i(t) are recorded:

a. The effect of α_1 and α_2 on the transient response are noted at t = 0.
b. For fixed values of c the output of i(t) decays to almost 0.
c. The rate of decay of i(t) depends on the values of R, α_1 and α_2. It has been observed that for the values of R = 1, c = 3, α_1= 0.6 and α_2 = 0.8, the current decays faster with less settling time. It has been observed that if value of R, α_1 and α_2 increased, the settling time also increases.

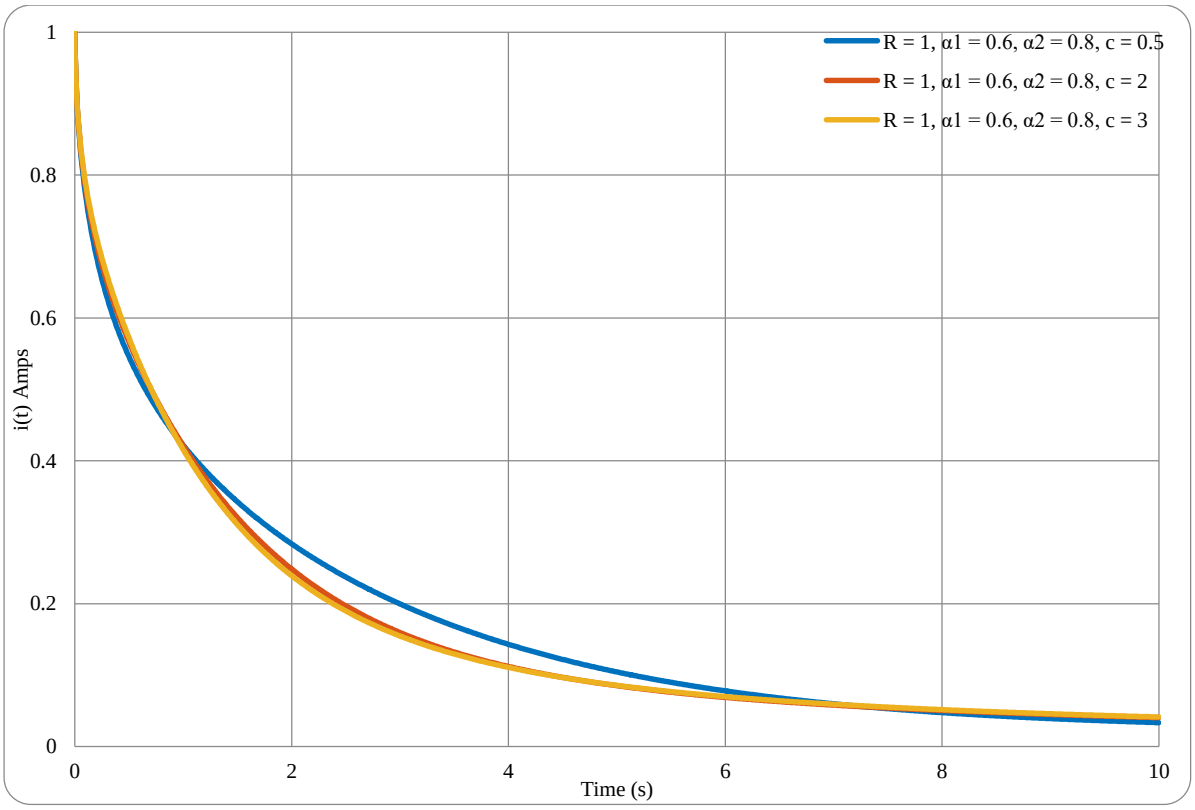

Fig. 8 Current evolution with respect to time considering initial condition as i_0 for homogeneous systems variable-order FDE, $\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)e^{-ct}$, considering α_1 and α_2 values constant and variations in the value of c

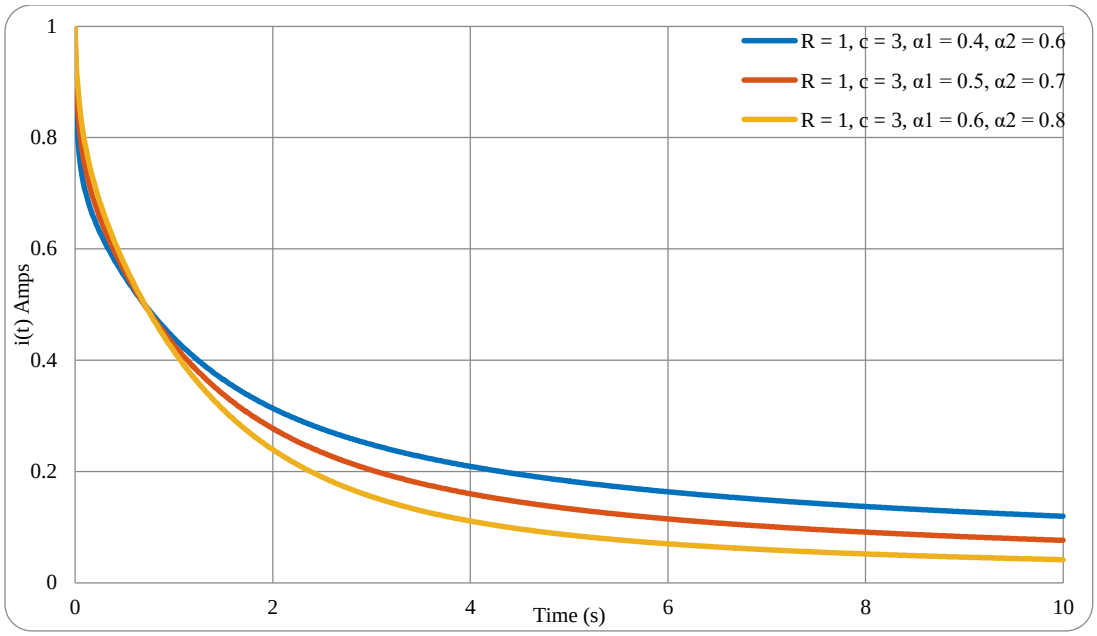

Fig. 9 Current evolution with respect to time considering initial condition as i_0 for homogeneous systems VOFDE, $\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)e^{-ct}$ with parameter c value constant and the variations in the values of α_1 and α_2

4.1.5. Case-5

$$\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)E_\beta(-ct^\beta)$$

here, $E_\beta(-ct^\beta)$is the Mittag-Leffler function.

$$A(s) = \frac{\alpha_1 s^\beta + \alpha_2 c}{s(s^\beta + c)},$$

$$\psi_\alpha(s) = s^{-\frac{\alpha_1 s^\beta + \alpha_2 c}{s(s^\beta + c)}}.$$

Substituting in Equation (18) gives

$$I(s) = \frac{i_0}{s\left(1 + RS^{-\frac{\alpha_1 s^\beta + \alpha_2 c}{s(s^\beta + c)}}\right)}, \quad (23)$$

Using the numerical ILT technique, the transient response with case-5 with Mittag-Leffler function and initial condition i_0 is obtained. To observe the influence of α(t) on the circuit behaviour, the parameters listed in Table 6 are selected for further analysis.

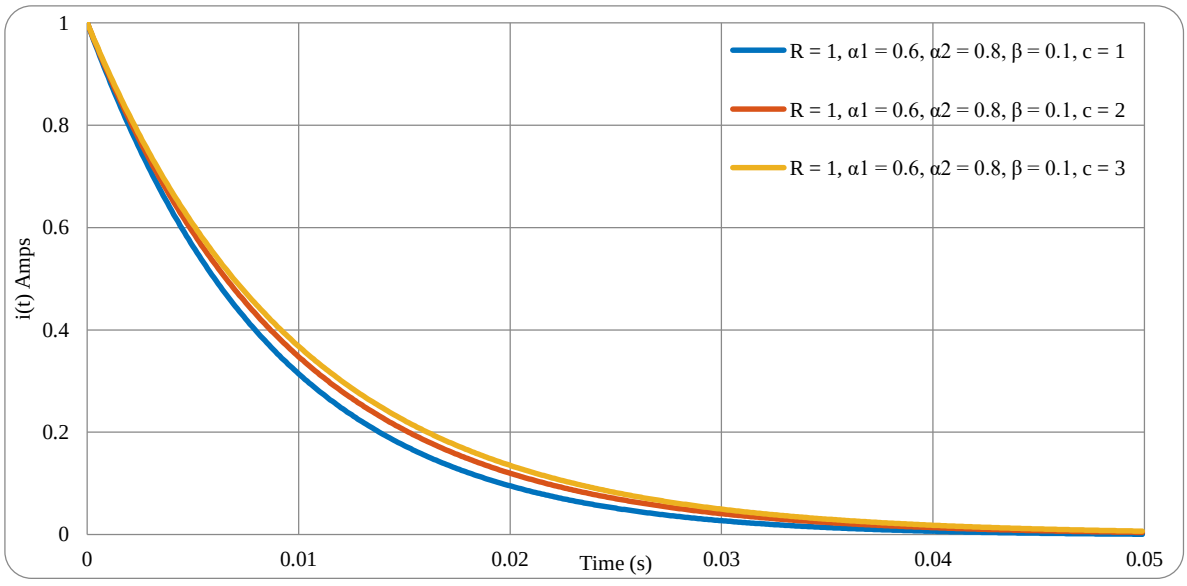

Fig. 10 Current evolution with respect to time considering initial condition as i_0 for homogeneous systems VOFDE $\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)E_\beta(-ct^\beta)$ with β value constant and the value of c varied

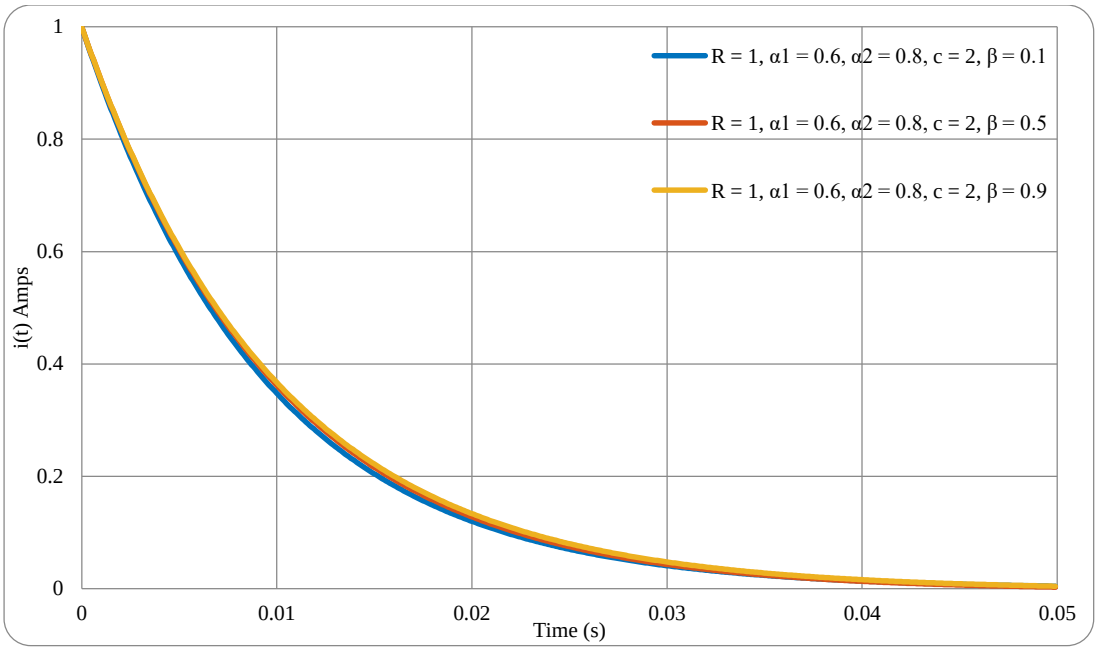

Fig. 11 Current evolution with respect to time considering initial condition as i_0 for homogeneous systems VOFDE $\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)\, E_\beta\,(-ct^\beta)$ for constant value of c and variations the value of β

Table 6. Variations in parameters α_1, α_2, β and c

Sr. No	α_1	α_2	c	β
1	0.6	0.8	1	0.1
2	0.6	0.8	2	0.1
3	0.6	0.8	3	0.1
4	0.4	0.6	2	0.1
5	0.5	0.7	2	0.5
6	0.6	0.8	2	0.9

The values of $R = 1$ is constant, and the response is shown for different values of α_1, α_2, β and c in Figures 10 and 11.

The following observations are examined.

a. The effect of parameters α_1, α_2, β and c on the transient response are noted at t = 0.
b. Almost for all values of β and c, the output of i(t) decays to almost 0.
c. The decay rate i(t) depends on the values of α_1 and α_2. It has been observed that for the values of R = 1, c = 3, α_1= 0.6 and α_2 = 0.8, the current decays faster with less settling time. It has been observed that if value of R, α_1 and α_2 increased, the settling time also increases.

4.2. Analysis of Series RL Circuit with Source

The application of circuit balance equations to circuit in Figure 2 results in following linear non-homogeneous FVODE.

$$ {}^{s}D_0^{\alpha(t)}\, i(t) + R\, i(t) = v(t), \tag{24}$$

Assuming v(t) a DC source with v = 1 volt. Substituting the parameter values, assuming initially relaxed circuit, the Laplace transform of Equation (24) is,

$$ s^{s\,A(s)} + I(s) - s^{s\,A(s)-1}\, i_0 + R\, i(s) = V(s), $$

With $\mathcal{L}v(t) = V(s) = \frac{1}{s}$,

Where, i(0) = i_0, $R > 0$, $R \in \mathbb{R}^+$

$$ \left(s^{sA(s)} + R\right) i(s) = s^{sA(s)-1} i_0 + V(s), $$

$$ I(s) = \frac{i_0}{s\left(1 + R\,\psi_\alpha(s)\right)} + \frac{V(s)}{\left(s^{sA(s)} + R\right)}. $$

Assuming R = 1, we get

$$ I(s) = \frac{1}{s\left(s^{sA(s)} + 1\right)}. \tag{25}$$

Again, the circuit transient analysis is presented for the $\alpha(t)$ expressions as given in Table 1.

4.2.1. Case-1

$\alpha(t) = \beta e^{-at}, \quad \beta \in (0,1) \quad$ and $\quad a \in \mathbb{R}^+$

$A(s) = \frac{\beta}{s+a}$, and

$\psi_\alpha(s) = \frac{-\beta s}{s+a}$.

Substituting in Equation (25) and employing the numerical ILT, the values of i(t) are computed. With the values of β and a as given in Table 7, the behavior of circuit current i(t) is shown in Figures 12 and 13.

Table 7. Variations in parameters β and a

Sr. No	β	a
1	0.1	1
2	0.1	5
3	0.1	10
4	0.5	10
5	0.9	10

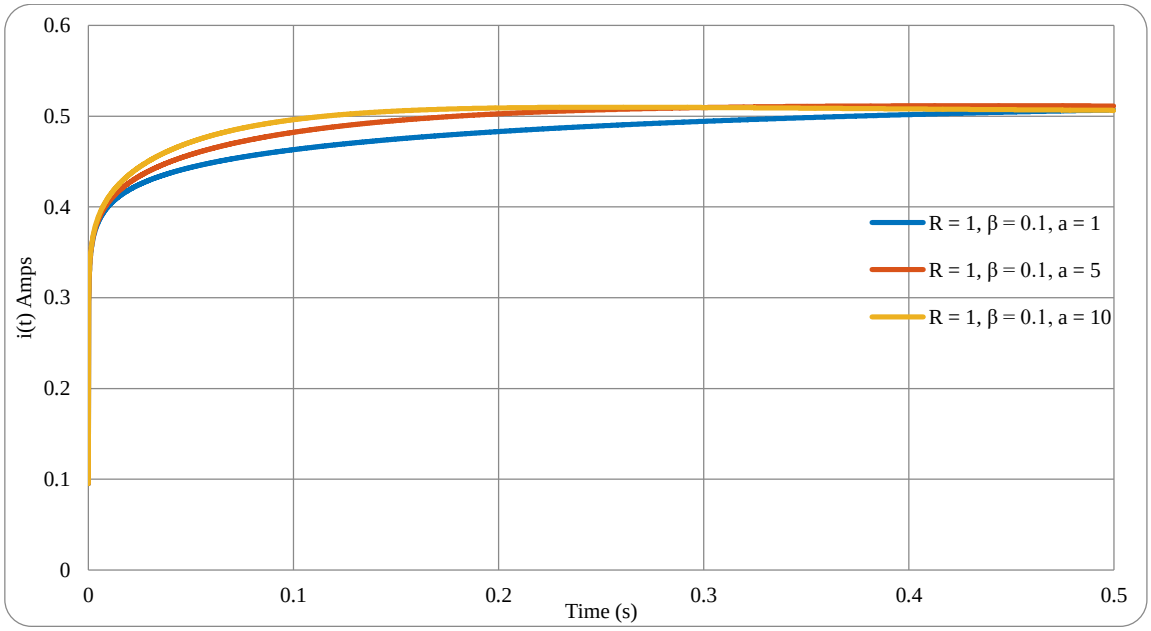

Fig. 12 Step response of non-homogeneous systems variable-order FDE $\alpha(t) = \beta e^{-at}$ considering β value constant and variations in the value of a

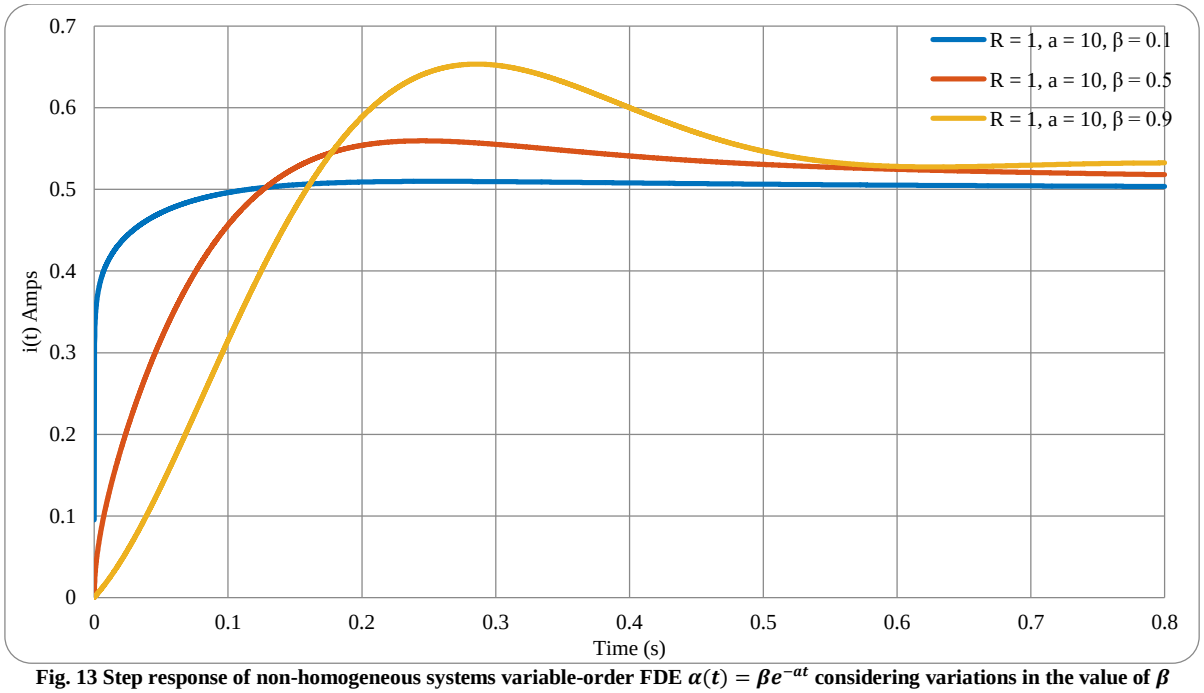

Fig. 13 Step response of non-homogeneous systems variable-order FDE $\alpha(t) = \beta e^{-at}$ considering variations in the value of β

From the response, it has been observed that:

a) The values of β and a influence the rise of i(t) in the transient period.
b) With β constant, and increasing value of a, the rate of rise of i(t) increases. However, the current finally settles at a steady-state value.
c) Substantial and noticeable change in transient behavior of i(t) is observed when β is increased, keeping value a constant. For β = 0.1, the current is overdamped. However, for higher values of β (0.5 and 0.9), the current exhibits an underdamped (oscillatory) behavior. A minor shift in the steady-state value of i(t) is also seen.

4.2.2. Case-2

$\alpha(t) = \beta(1 + \sin t)$,

Where $\beta \in (0,0.5)$ and $a \in \mathbb{R}^+$

$A(s) = \frac{\beta}{s} + \frac{\beta}{s^2+1}$,

$\psi_\alpha A(s) = s^{-\beta s(\frac{1}{s}+\frac{1}{s^2+1})}$.

Substituting these values of $A(s)$ and $\psi_\alpha A(s)$ in Equation (25), and using the numerical ILT technique, the transient response to initial condition $i_0 = 0$ and $V(s) = \frac{1}{s}$. Different values of β considered in the analysis.

Here, a fixed value of $R = 1$ is considered, and the value of β is β= 0.1, 0.3, 0.49. The transient response of i(t) is as shown in Figure 14.

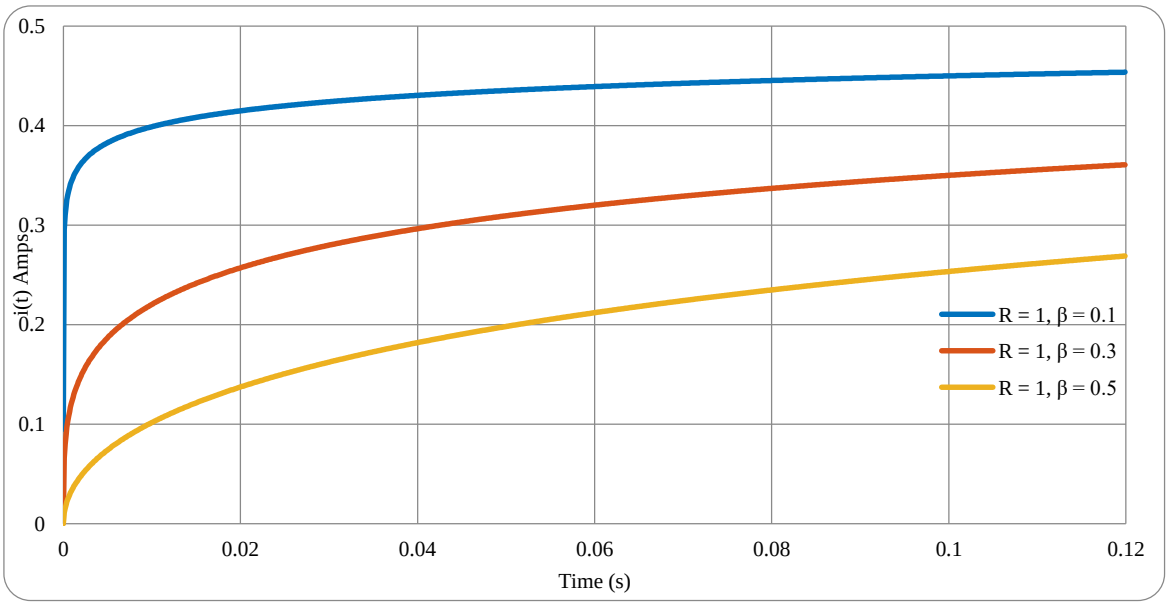

Fig. 14 Step response of non-homogeneous systems variable-order FDE $\alpha(t) = \beta(1 + \sin t)$, considering variations in the value of β.

From the response, it has been observed that,

a. At t = 0, the value of i(t) = i_0 = 0, for all cases.
b. The sinusoidal function present in $\alpha(t)$ does not force the current to follow sinusoidal behavior.
c. For the values of $R = 1$ and $\beta = 0.1$, the transient response is faster with less settling time and reduced oscillations.

4.2.3. Case-3

$\alpha(t) = \beta e^{-at}(1 + \sin t)$,

Here $\beta \in (0,0.5)$ and $a > 0$.

$A(s) = \beta\left[\frac{1}{s+a} + \frac{1}{(s+a)^2+1}\right]$,

$\psi_\alpha(s) = -\beta s\left[\frac{1}{s+a} + \frac{1}{(s+a)^2+1}\right]$.

Substituting the values of $A(s)$ and $\psi_\alpha A(s)$ in Equation (25), and using the numerical ILT technique, the transient response to initial condition $i_0 = 0$ and $V(s) = \frac{1}{s}$. The variations of parameter β and a considered in the analysis. To observe the influence of α(t) on the circuit behaviour, the parameters listed in Table 8 are selected for further analysis.

Table 8. Variations in parameters β and a

Sr. No	β	a
1	0.1	1
2	0.1	5
3	0.1	10
4	0.3	10
5	0.45	10

The behavior of circuit current i(t) is shown in Figures 15 and 16.

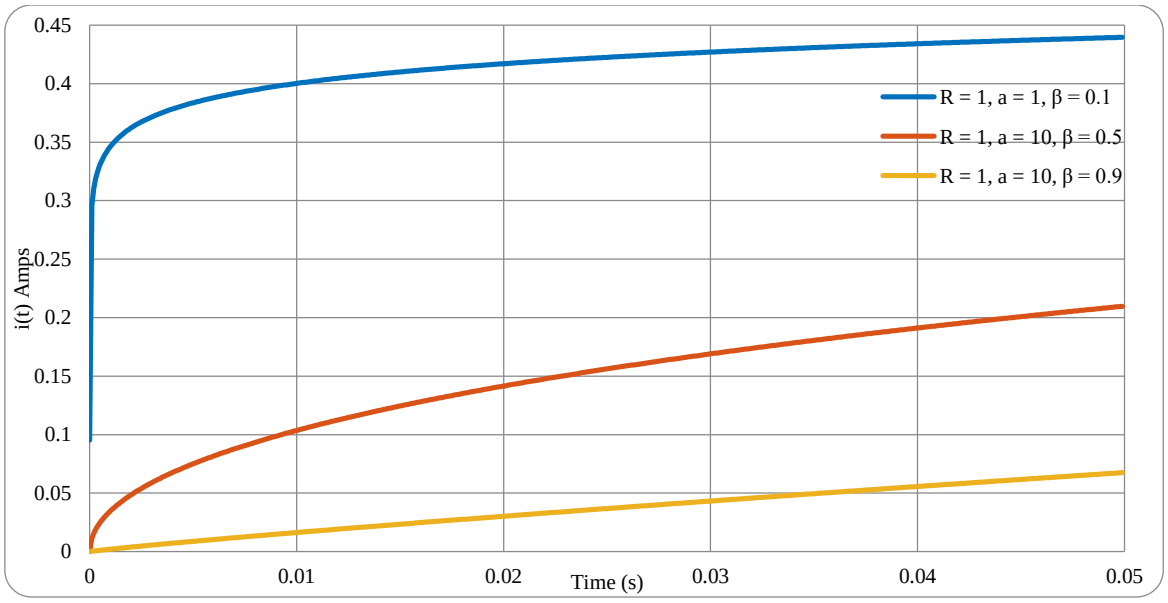

Fig. 15 Step response of non-homogeneous systems variable-order FDE $\alpha(t) = \beta e^{-at}(1 + \sin t)$ considering variations in parameter β

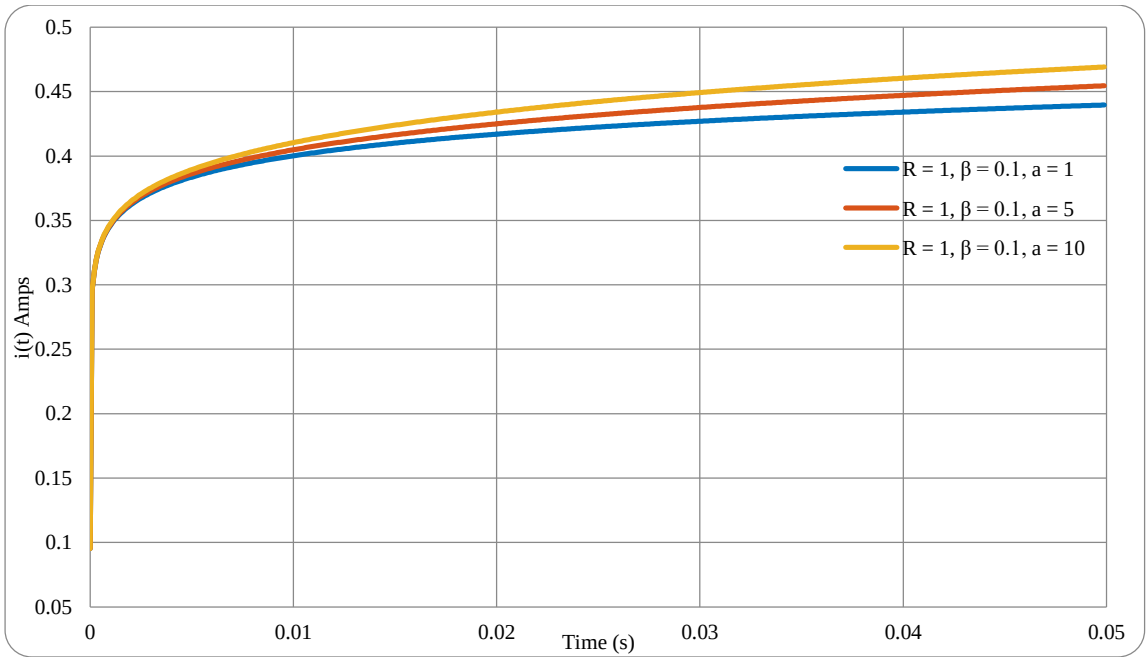

Fig. 16 Step response of non-homogeneous systems variable-order FDE $\alpha(t) = \beta e^{-at}(1 + \sin t)$ considering variations in parameter a

Following operations are recorded.

a. The parameters β and a significantly affect how i(t) evolves during the transient phase, at t = 0, the value of i(t) = i_0 = 0.
b. In contrast, varying β while keeping parameter a constant, produces more noticeable changes in the transient response. For β = 0.1, the response is overdamped. The current i(t) contains the transient as sin t is a part of the function $\alpha(t)$.
c. For $R = 1, \beta = 0.1$, and $a = 1$, the current settles fast with less settling time.

4.2.4. Case-4

$$\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)e^{-ct},$$

$$0 < \alpha_1 < \alpha_2 < 1, \quad c > 0,$$

Where, $\alpha_1 = 0.6, \quad \alpha_2 = 0.8, \quad c > 0$

$A(s) = \frac{\alpha_2 c + \alpha_1 s}{s(s+c)}$, and

$\psi_\alpha(s) = s^{-\frac{\alpha_2 c + \alpha_1 s}{c+s}}$,

$i(0) = i_0$, and for forcing function $v(t) = 1$,

We have, $V(s) = \frac{1}{s}$, so

$$i(s) = \frac{i_0}{s+(1+R\psi_\alpha(s))} + \frac{\frac{1}{s}}{s^{sA(s)}+R}. \quad (26)$$

To evaluate Equation (26), substituting the value of $\psi_\alpha(s)$and A(s) and using the numerical ILT technique, the transient response with case-4 and initial condition i_0 is obtained. To observe the effect of α(t) on the circuit behaviour, the parameters listed in Table 9 are selected for further analysis.

Table 9. Variations in parameters α_1, α_2 and c

Sr. No	α_1	α_2	c
1	0.6	0.8	1
2	0.6	0.8	2
3	0.6	0.8	3
4	0.4	0.6	3
5	0.5	0.7	3
6	0.6	0.8	3

The values of $R = 1$ is constant, and the response is shown for different values of α_1, α_2 and c in Figures 17 and 18.

Following observations are noted for the response:

a. The effect of α_1 and α_2 on the step response are noted at t = 0.

b. For fixed values of α_1 and α_2, and variations in c, there is no prominent change in the response in output of i(t).

c. The rate of settling time of current i(t) varies with variation in the values of α_1 and α_2. It has been observed that for the values of R = 1, c = 3, α_1= 0.6 and α_2 = 0.8, the current settles faster.

d. Thus, the power of the exponential function greatly affects the transient behavior of the circuit current.

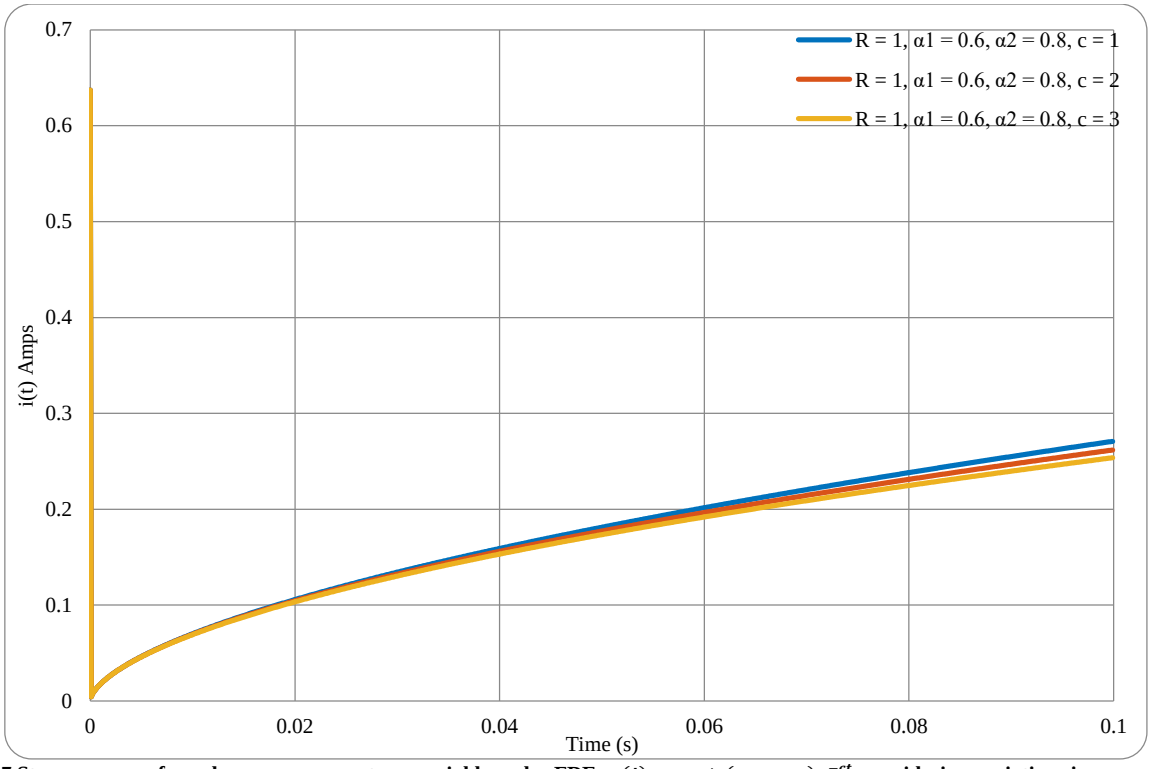

Fig. 17 Step response of non-homogeneous systems variable-order FDE, $\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)e^{-ct}$ considering variations in parameter c

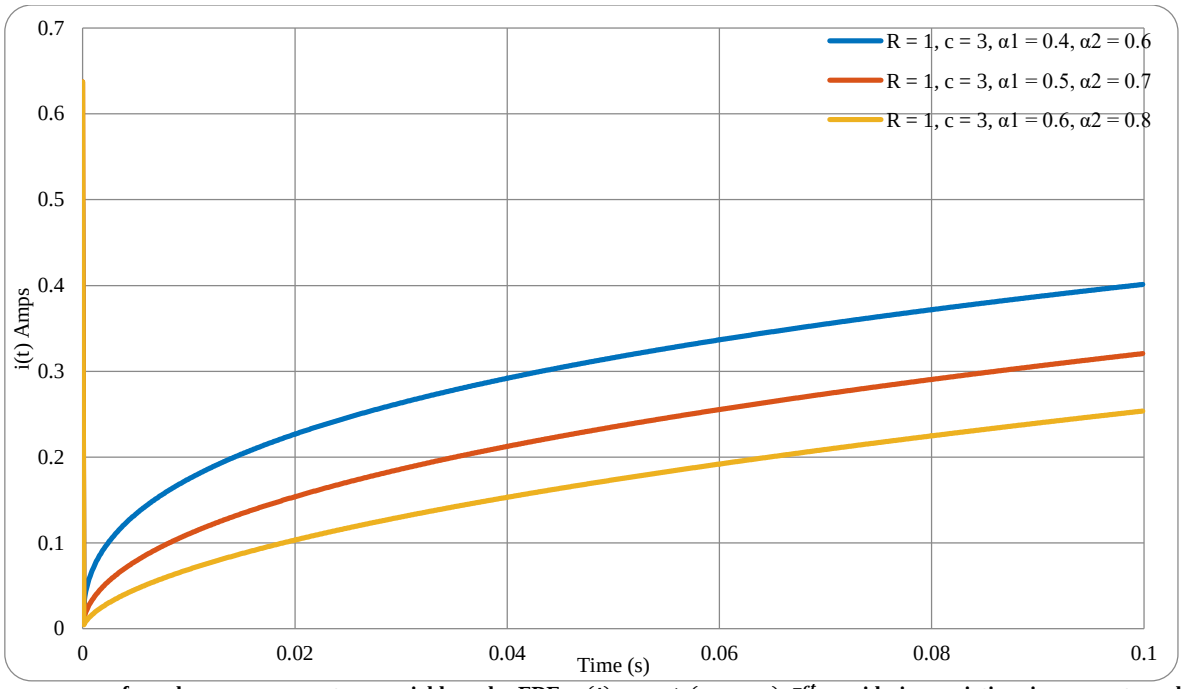

Fig. 18 Step response of non-homogeneous systems variable-order FDE $\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)e^{-ct}$ considering variations in parameter value α_1 and value α_2

4.2.5. Case-5

$$\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)E_\beta(-ct^\beta)$$

This case is generalization of case-4 with exponential function replaced with Mittag-Leffler function (MLF).

For this case, R = 100 ohm is considered.

$$A(s) = \frac{\alpha_1 s^\beta + \alpha_2 c}{s(s^\beta + c)},$$

$$\psi_\alpha(s) = s^{-\frac{\alpha_1 s^\beta + \alpha_2 c}{s(s^\beta + c)}}$$

To evaluate Equation (26), substitute the value of $\psi_\alpha(s)$ and use the numerical ILT technique, the transient response with case-5 and the initial condition $i_0 = 0$ is obtained.

To observe the influence of α(t) on the circuit behaviour, the parameters listed in Table 10 are selected for further analysis.

Table 10. Variations in parameters α_1, α_2, β and c

Sr. No	R	α_1	α_2	c	β
1	100	0.6	0.8	1	0.1
2	100	0.6	0.8	2	0.1
3	100	0.6	0.8	3	0.1
4	100	0.6	0.8	3	0.1
5	100	0.6	0.8	3	0.5
6	100	0.6	0.8	3	0.9

The values of $R = 100$, α_1, α_2 is constant, and the response is shown for different values of β and c in Figures 19 and 20.

Following observations are recorded for the response:

a. The effect of β and c on the step response are noted at t = 0 by keeping R, α_1 and α_2 constant.
b. For fixed values of R, α_1 and α_2, and for β = 0.1 and c = 3 the current i(t) settles fast.
c. Thus, the ML function greatly affects the transient behavior of the circuit current.

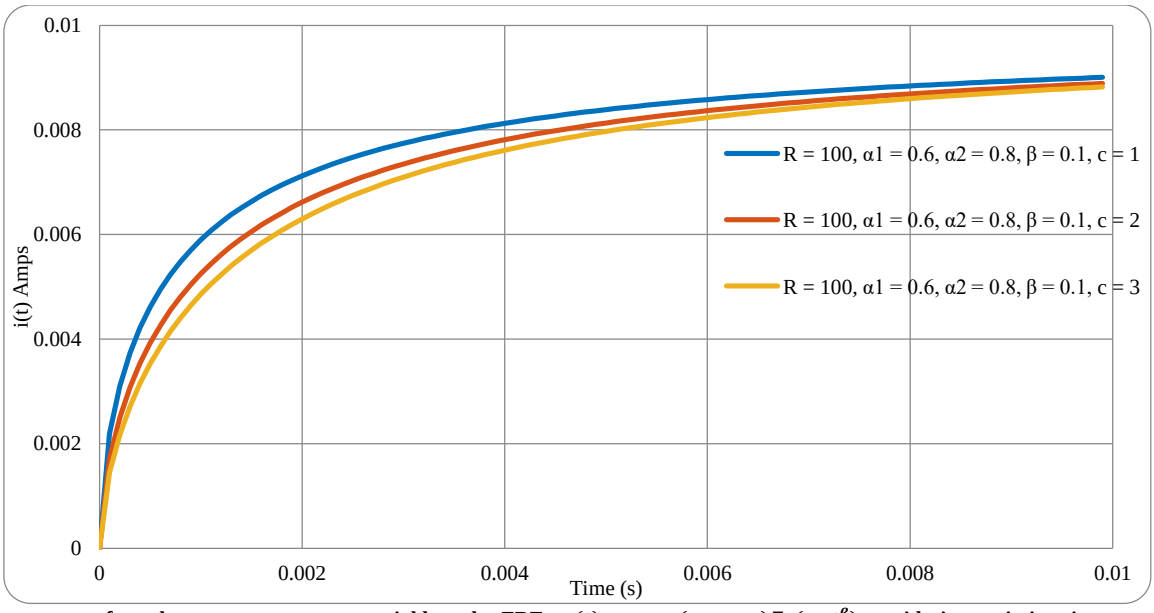

Fig. 19 Step response of non-homogeneous systems variable-order FDE, $\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)E_\beta(-ct^\beta)$ considering variations in parameter value c

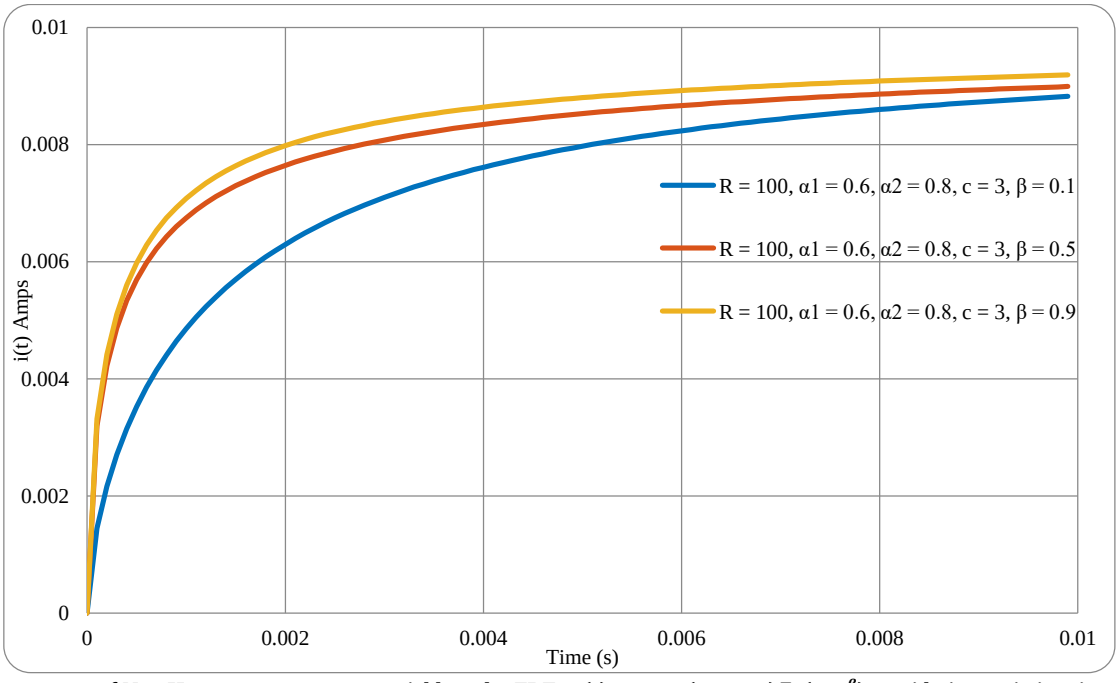

Fig. 20 Step response of Non-Homogeneous systems variable-order FDE, $\alpha(t) = \alpha_2 + (\alpha_1 - \alpha_2)E_\beta(-ct^\beta)$ considering variations in parameter β

5. Discussion

The study presented in the previous section highlights:

a) The dominant effect of the choice of VO derivative function $\alpha(t)$ on the dynamics of both circuits.
b) In the RL circuit, in general, the decay/rise rate of the circuit current is generated by the value of resistance (R). A higher value of R allows a fast dissipation of heat, thereby forcing a faster decay of inductor energy.
c) In the case of inductor modelled using VO dynamics, the choice of the $\alpha(t)$ and its parameters dictates the speed of decay/rise of the current.
d) The constitution of the $\alpha(t)$(presence of mathematical terms in $\alpha(t)$) clearly distinguishes the circuit dynamics for each case.

Furthermore, the following can be concluded regarding the structure of $\alpha(t)$:

i) The presence of exponential and MLF terms in $\alpha(t)$ greatly influences the transient behavior of i(t).

ii) The presence of sinusoidal terms does not dominate the transient in i(t).
iii) The choice of $\alpha(t)$ also affects the steady state value at which the i(t) settles.
iv) Especially, the presence of sinusoidal term in $\alpha(t)$ (see case-2 and case-3) does not allow the current to settle to zero value at steady-state in case of series RL circuit without a source. This is an interesting conclusion and is worth exploring further. It seems that the presence of sinusoidal function in $\alpha(t)$ somehow retains the energy in the inductor for a long time.

Overall, the study presents a rigorous analysis of series RL circuits wherein the inductor dynamics is modelled using a VO derivative. The choice of a particular $\alpha(t)$ to model the inductor can be decided depending on its type, construction, and application.

6. Conclusion

The modelling of linear circuits using scarpi definition of variable-order derivatives and their analysis is presented in this work. The scarpi definition of VO derivatives and integrals form Sonine pair and are a mathematically sound extension of fractional-order differintegrals. The study presented the effect of changing the parameters of VO derivative order on the circuit dynamics. The study confirms that the nature and parameter values of the variable-order greatly affect the transient and steady-state response of the circuit under consideration. The transient behaviour showcases substantial variation, from underdamped to overdamped response. The steady-state value at which the current settles is also influenced by the nature of variable-order. The study highlights a significant step in the analysis of linear VO circuits. The results presented in this work can be extended by analyzing the complicated linear VO circuits and investigating the non-linear VO circuits.

Conflicts of Interest

The authors have no conflict of interest and competing interests to declare that are relevant to the content of this article.

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