

An Analysis of the effect of the peripheral viscosity on bile flow characteristics through cystic duct with stone: Study of two-layer model with squeezing

Suchi Agarwal^{#1}, S.P.Singh²

^{1,2}Dayalbagh Educational Institute, Agra, India

Abstract - Bile flow in the human biliary system plays a vital role in the pathogenesis of biliary diseases. In the present study, the effect of an overlapping stone in the cystic duct on the flow characteristics of bile is observed. The mathematical model here, considers a two-fluid model of bile, consisting of a core region of suspension of bile salts, cholesterol and bilirubin and peripheral layer of mucosa. The expression for the flow characteristics like the resistance, the wall shear stress, and the stress at the throat and at the critical height of stone is studied. It is observed that as the peripheral viscosity decreases the resistance to flow and the shear stress also decreases.

Keywords — Bile, Cystic duct, Gallstone, Shear stress, Gallstone.

I. INTRODUCTION

The presence of gallstone is one of the most common human biliary diseases which is known as Cholelithiasis. Clinical treatments for the diseases often involve the surgical removal of the gallbladder, known as cholecystectomy, which is the most often performed abdominal operation in the West. In order to understand the cause of biliary diseases, it is important to understand the mechanical behaviour of the human biliary system. The role of human biliary system is to create, store, transport and release bile into the duodenum to aid digestion of fats Fig .1. It consists of an organ and duct system. Bile, the liquid that flow in the biliary system is a yellow-brown fluid produced by the liver and is composed of three main components –cholesterol, bile salts and bilirubin. When the gallbladder is not functioning properly, components of bile are supersaturated leading to the formation of solid crystals called gallstones. Most common biliary diseases are:

- Cholelithiasis - the presence of gallstones.
- Cholecystitis – the inflammation of gallbladder.

Jungst et al¹ suggested that the super saturation and rapid nucleation of cholesterol in bile were of key importance in the pathogenesis of cholesterol

gallstones. Ooi et al² presented a numerical study of steady flow in human cystic duct models. Idealized models were constructed, first with staggered baffles in a channel to represent the valves of Heister and lumen. Al-Atabi et al³ presented a mathematical model accounting for the effects of geometry of the baffle configurations. It predicted the pressure drop in circular pipe fitted with segmental baffles. Bird et al⁴ described the variation in geometry of the cystic duct, obtained from acrylic resin casts of the neck and first part of the cystic duct in gallbladders removed for gallstones disease and obtained from patients undergoing partial hepatectomy for metastatic disease. Li et al⁵ developed a mechanical model for the human biliary system during the emptying phase, based on a clinical test in which gallbladder volume changes were measured in response to a standard stimulus and a recorded pain profile.

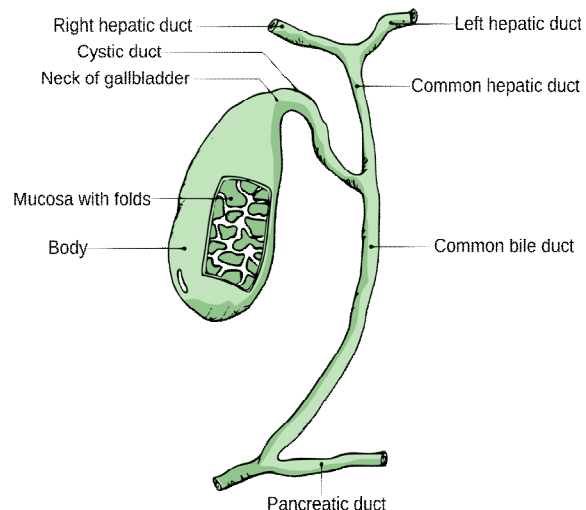


Fig.1 Gallbladder and Cystic Duct

Li et al⁶ worked on two one-dimensional models to estimate the pressure drop in the normal human biliary system for Reynolds number up to 20.

Excessive pressure drop during bile emptying and refilling may result in incomplete bile emptying, leading to bile stasis and subsequent gallstones formation. The effects of biliary system geometry, elastic property of cystic duct and bile viscosity on pressure drop was studied. It was found that the maximum pressure drop occurs during bile emptying immediately after a meal, and was greatly influenced by the viscosity of bile and the geometric configuration of the cystic duct. Luo et al⁷ researched to understand the physiological and pathological functions of the biliary system. It was believed that the mechanical factors play an important role in the mechanism of the gallstone formation and biliary diseases.

Li et al⁸ extended their previous study of the human biliary system to include two new factors :the non-Newtonian properties of bile, and elastic deformation of the cystic duct. A one – dimensional (1D) model was analysed and compared with three– dimensional fluid structure interaction simulations. It was found that non – Newtonian bile raises resistance to the flow of bile, which can be increased enormously by the elastic deformation(collapse) of the cystic duct.

Agarwal et al¹¹ presented a theoretical investigation of the characteristics of bile flow through a cystic duct with stones taking bile as Hershel -Buckley fluid. Agarwal et al¹² studied taking bile as a Casson fluid. The expression for the flow characteristics like the resistance and the wall shear stress was studied. Agarwal et al¹³ extended their study further taking bile as a non-Newtonian fluid and power law fluid model is taken in account to study the flow of bile in the duct.

II.FORMULATION AND SOLUTION OF THE PROBLEM

Consider the two-layer model of bile, consisting of a core region of suspension of bile salts, cholesterol and bilirubin and peripheral layer of mucosa. The axially symmetric flow of bile in the uniform cystic duct of radius R with axially symmetric overlapping stone. The flowing bile is laminar, steady and axially symmetric. The rigid cystic duct is having axially symmetric stones and the role of valves of heister is considered to be negligible. The maximum height of the stone is much less as compared to the length and the uniform radius of the cystic duct. The problem is studied under the following assumptions -

1. The motion is slow, so the inertia effect can be neglected.
2. Flow is incompressible.

Further, in the present study the effect of axially symmetric overlapping stone on the core fluid and peripheral mucosa is investigated for parameters like flow resistance and wall shear stress.

In view of the above considerations the basic equation governing the flow of bile in the rigid cystic duct is given by:-

Governing equation

The general equation for the flow of bile is (Shukla et al¹⁰)

$$-\frac{dP}{dz} + \frac{1}{r} \frac{\partial}{\partial r} \left[\mu(r)r \frac{\partial u_z}{\partial r} \right] = 0$$

(1)

The consistency function $\mu(r)$ is taken as

$$\mu(r) = \mu_1 \quad \text{for} \quad 0 \leq r \leq R_1(z)$$

$$\mu(r) = \mu_2 \quad \text{for} \quad R_1(z) \leq r \leq R(z)$$

(2)

In equation (1) and (2), z is the co-ordinate along the axis of the cystic duct in the flow direction, r is the co-ordinate in the radial direction and perpendicular to flow, $\mu(r)$ represents the viscosity function; P is the

pressure at any point, $\frac{dP}{dz}$ is the pressure gradient, u_z

stands for the axial velocity of fluid, μ_1 and μ_2 are the value of viscosity function at the central core region and the peripheral region. Also $R_1(z)$ and $R(z)$ are the radius of the central core region and of the duct in the constricted region.

Boundary conditions

The equation (1) is solved using the following boundary condition:

(i)

$$\frac{\partial u_z}{\partial r} = 0 \quad \text{at} \quad r = 0$$

(3)

(ii)

$$u_z = 0 \quad \text{at} \quad r = R(z)$$

(4)

(iii)

$$P = P_0 \quad \text{at} \quad z = 0 \quad \text{and} \quad P = P_L \quad \text{at} \quad z = L$$

(5)

(iv)

$$u_2 = u_1 \quad \text{at} \quad r = R_1(z)$$

(5a)

The equation (4) and (5a) are known as no-slip condition and matching condition respectively.

Geometry Used

The overlapping geometry of the cystic duct and the shape of the central layer in Fig.2 and Fig.3 respectively is written as –

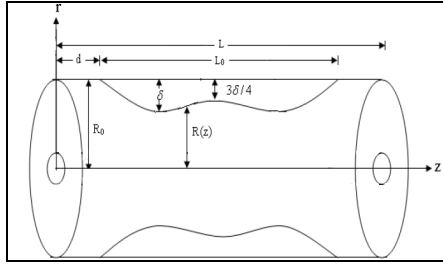


Fig.2 The overlapping geometry

$$\frac{R(z)}{R_0} = 1 - \frac{3}{2} \frac{\delta}{R_0} \frac{1}{L_0^4} \begin{bmatrix} 11(z-d)L_0^3 \\ -47(z-d)^2 L_0^2 \\ +72(z-d)^3 L_0 \\ -36(z-d)^4 \end{bmatrix}$$

$$d \leq z \leq d + L_0$$

=1, Otherwise

(6)

and

$$\frac{R_1(z)}{R_0} = \alpha - \frac{3}{2} \frac{\delta_1}{R_0} \frac{1}{L_0^4} \begin{bmatrix} 11(z-d)L_0^3 - \\ 47(z-d)^2 L_0^2 \\ +72(z-d)^3 L_0 \\ -36(z-d)^4 \end{bmatrix}$$

$$, d \leq z \leq d + L_0$$

=α, Otherwise

(7)

Where R_0 , the radius of the duct without stones, L_0 is the length of stone (constricted region), L is the duct length, d indicates the location of stone, α is the radius of the central core radius to tube radius in the unobstructed region and (δ, δ_1) are the maximum height of and bulging at two locations i.e. at $z = d + \frac{L_0}{6}$ and $z = d + \frac{5L_0}{6}$. (V.P.Srivastav et al)⁹

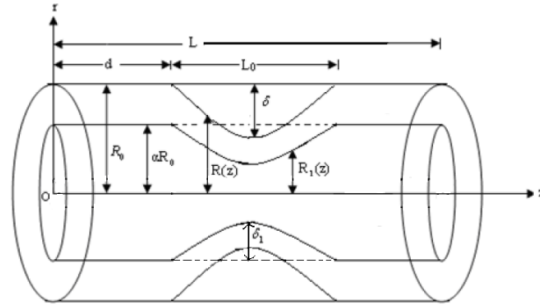


Fig.3 The shape of the central layer. The inner layer comprises of bile salts, bilirubin and cholesterol and outer layer is of mucus.

Analysis

On integrating (1) and using (3) we get:

$$\frac{\partial u_z}{\partial r} = \frac{r}{2\mu(r)} \frac{dP}{dz}$$

The flow flux, Q is defined as

$$Q = \pi \int_0^R r^2 \left(-\frac{\partial u_z}{\partial r}\right) dr \tag{8}$$

From equation of continuity it can be noted that Q is a constant.

Now using (8) and (2) we get the total flux Q as

$$Q = \frac{\pi P}{8\mu_2} (R^4(z) - (1-\mu)R_1^4(z)) \tag{9}$$

Where, $\mu = \frac{\mu_2}{\mu_1}$

From equation (9) and (5), the pressure gradient is written as follows:

$$P_0 - P_L = \frac{8Q\mu_2}{\pi} \left[\int_0^L \frac{dz}{R^4(z) - (1-\mu)R_1^4(z)} \right] \tag{10}$$

Thus the resistance to flow λ is defined

$$\lambda = \frac{P_0 - P_L}{Q} = \frac{8\mu_2}{\pi R_0^4} \left[\int_0^L \frac{dz}{\left[\left(\frac{R}{R_0}\right)^4 - (1-\mu)\left(\frac{R_1}{R_0}\right)^4\right]} \right] \tag{11}$$

Therefore, in dimensionless form the flow resistance may be expressed as $\bar{\lambda}$, where

$$\bar{\lambda} = \frac{\mu}{\beta} \left[1 - \frac{L_0}{L} + \frac{1}{L} \int_d^{d+L_0} \frac{dz}{(R/R_0)^4} \right] \quad (12)$$

$$\beta = 1 - (1 - \mu)\alpha^4 \quad (13)$$

The above expression gives the flow resistance in the cystic duct.

Shearing Stress

The shearing stress at the wall using (9) can be defined as -

$$\tau_r = \frac{4Q\mu_2}{\pi\beta R_0^3 \left(\frac{R}{R_0}\right)^3}$$

(14)

The non-dimensional shearing stress at the wall is given by

$$\tau_w = \frac{\tau_r}{\tau_N} = \frac{\mu}{\beta(R/R_0)^3}$$

(15)

The non-dimensional shearing stress at stone throat is given by

$$\tau_1 = \frac{\mu}{(1 - (1 - \mu)\alpha^4) \left(1 - \frac{5}{4} \frac{\delta}{R_0}\right)^3} \quad (16)$$

Similarly, the non-dimensional shearing stress at the critical height of stone is-

$$\tau_2 = \frac{\mu}{\beta \left(1 - \frac{3}{4} \frac{\delta}{R_0}\right)^3} \quad (17)$$

The expression (15), (16) and (17) exhibits the non-dimensional values of shearing stress at the wall, at the throat of constriction and at the critical height of constriction (stone).

Results and Discussion

The graphs have been plotted for the above expressions (12), (15), (16) and (17). Fig.3a shows the variation of impedance with the changing stone height with varying viscosity and parameter α . It is observed that the flow resistance increases with increasing height of stones. Also the resistance to flow decrease as viscosity μ decreases (when peripheral viscosity decreases). For any given parameter μ , the flow characteristics $\bar{\lambda}$ decreases with decreasing values of the parameter α . Fig.3b, exhibits the variations of flow resistance with stone height for different values of the length of the stone. It is

observed that as the stone length increases, the value of $\bar{\lambda}$ also increases. Fig.4a, 4b and 4c demonstrates the variation of wall shear stress (dyne/cm²) with the axial distance z(cm) with different parameters in the region of stone length. It shows that the wall shear stress increases with the increasing viscosity. At the axial distance in the region of stone, τ_w decreases with the decreasing value of the parameter α , but increases with increasing value of the parameter δ/R_0 . In the above case, the wall shear stress τ_w in the non-uniform region, rapidly increase from its values at z=0 to the highest value in the up

flow of the first throat at $z = d + \frac{L_0}{6}$, it then decrease

sharply in the down flow of the first throat, occupying minimum value at the critical height. The wall shear stress increases steeply in the upstream of the second stone throat and attains its highest magnitude (with the same value as at the first throat of the stones) at the

second throat of the stone, at $z = d + \frac{5L_0}{6}$, it then

decreases rapidly to the same magnitude as its value at z=0 (i.e. at the end point of the constriction profile). It is also observed that the flow characteristic, τ_w

assumes significantly lower magnitude at critical height of the stone than at the throat of the stone. Fig.5, illustrates the comparison of variation in wall shear stress with stone height at the throat of constriction and at the critical height for viscosity μ . It clearly shows the decrease in the value of shear stress with the decrease in viscosity. Moreover, the magnitude of stress is considerably high at the throat of stone than at the critical height. On the other hand, stress increases as the stone height increases. Fig.6a and 6b, illustrates the variation in the shear stress at the critical height with stone height for different parameters. The shear stress decreases with the decreasing value of the viscosity and parameter α . As the stone height increases the shear stress also increases.

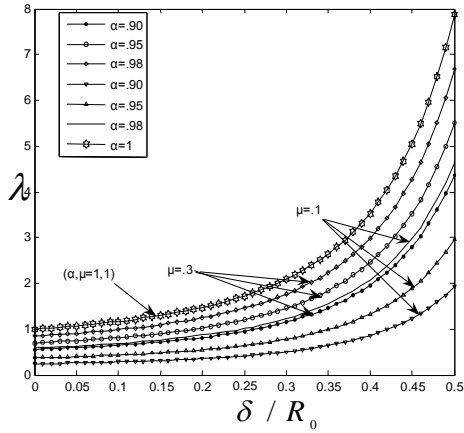


Fig.3a: Variation of resistance to flow with stone height for viscosity and parameter α .

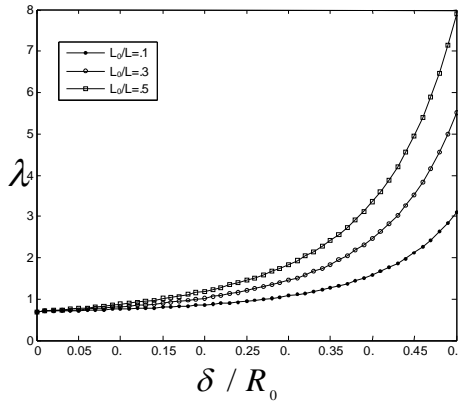


Fig.3b: Variation of resistance to flow with stone height for stone length

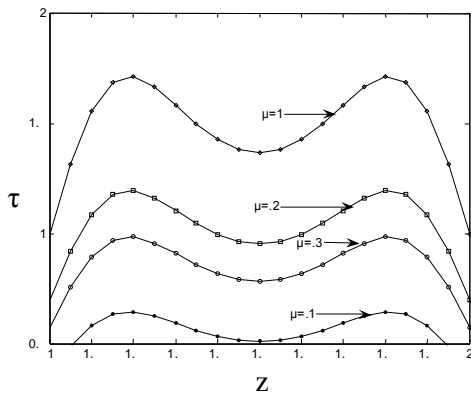


Fig.4a : Variation of wall shear stress with axial distance for viscosity.

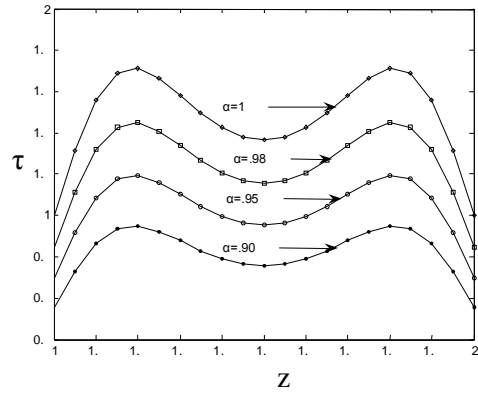


Fig.4b: Variation of wall shear stress with axial distance for α .

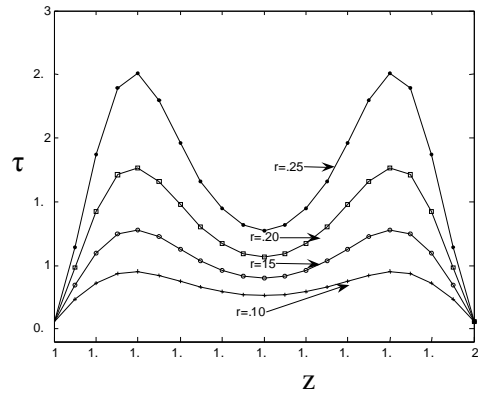


Fig.4c: Variation of wall shear stress with axial distance for stone height.

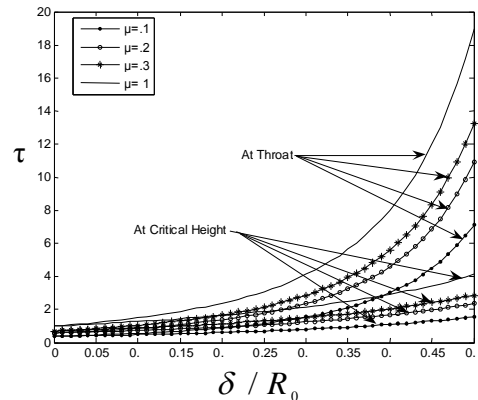


Fig.5: Comparative variation of wall shear stress with stone height for viscosity μ for critical height and at throat of stone.

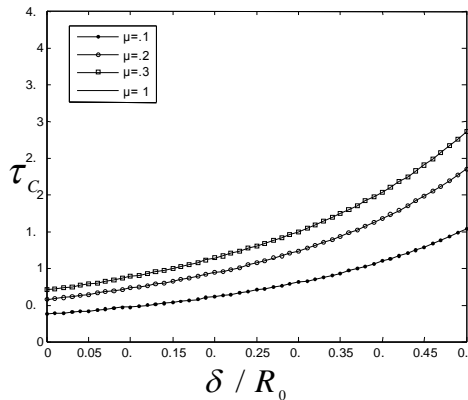


Fig.6a: Variation of wall shear stress with stone height for viscosity μ at critical height of stone.

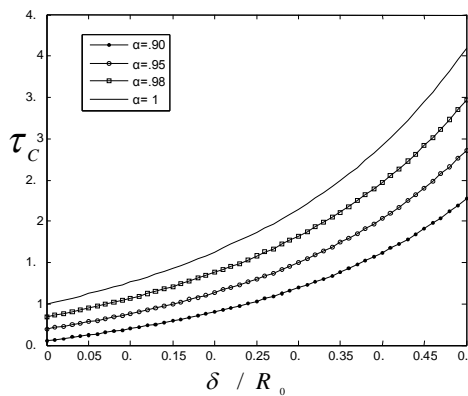


Fig.6b: Variation of wall shear stress with stone height for α at critical height of stone.

CONCLUSION

The present study deals with a theoretical investigation of the characteristics of bile through a diseased cystic duct. Using appropriate boundary conditions, analytical expressions of the shear stress and the flow resistance have been obtained. The computed results are presented graphically for different values of the parameters μ , α and δ / R_0 that stand respectively for viscosity, ratio of central core radius to tube radius and the stone height. The effect of peripheral layer viscosity on the flow characteristics of bile in the presence of overlapping stone in the cystic duct has been studied. The flow characteristics i.e. resistance and shear stress are maximum at the throat of stone than at the onset of the stone. The shear stress at the critical height of the stone assumes significantly smaller value than at the stone throat. Moreover, maximum increase in the flow characteristics is observed when the viscosity in core region and in the mucus exhibit similar magnitude.

The migration of a gallstone into the opening of the cystic duct may block the outflow of bile during

gallbladder contraction. Cystic duct obstruction, if it persists for more than a few hours, may lead to acute gallbladder inflammation.

This analysis indicates that the peripheral layer of mucous helps in the functioning of the diseased cystic duct. The motivation behind the proposed study is to develop a mathematical model of the flow of bile and influence of factors affecting it, and to understand the role of fluid dynamics in human biliary system, so that it can be used in prescribing preventive measures to patients at risk to gallstones.

REFERENCES

- [1] Jungst D, Lang T, Huber P, Lang V and Paumgartner G. Effect of phospholipids and bile acids on cholesterol nucleation time and vesicular/micellar cholesterol in gallbladder bile of patients with cholesterol stones. *Journal of lipid research*. Vol.34, 1993.
- [2] Ooi R, Luo XY, Chin S, Johnson A and Bird NC. The flow of bile in the human cystic duct. *Journal of biomechanics*. 37: 1913-1922.(2004)
- [3] Al-Atabi M, Chin SB, Al-Zuhair S and Luo XY. Pressure drop in laminar and turbulent flows in circular pipe with baffles – An experimental and analytical study. *International Journal of Fluid Mechanics Research*. Vol. 33, No. 4,(2006).
- [4] Bird N C, Ooi R, Chin SB and Johnson A. Investigation of the functional three-dimensional anatomy of the human cystic duct: A single helix. *ClinAnat*. 19(6); 528-534, 2006.
- [5] Li W G, Luo XY, Bird N and Johnson A. Correlation of mechanical factors and gallbladder pain., *J of Computational and Mathematical methods in Medicine*.
- [6] Li W, Chin SB, Johnson A and Bird NC. One dimensional models of the human biliary system. *J of Biomechanics*. Vol.129, 2007.
- [7] Luo XY, Li W, Bird N, Chin SB and Johnson A. On the mechanical behavior of the human biliary system. *World J. Gastroentero*. March 7; 13(9):1384- 1392(2007).
- [8] Li W, Luo XY, Chin SB, Hill NA, Johnson A and Bird NC. Non-Newtonian bile flow in elastic cystic duct: one and three dimensional modeling. *Annals of Biomedical Engineering*, Vol.36, No.11, 2008.
- [9] Srivastava V P, Vishnoi R, Mishra S and Sinha P. A two layered Non-Newtonian arterial blood flow through an overlapping constriction. *Applications and Applied Mathematics: An International Journal*. Vol.6, Issue 11(june 2011)pp. 1781-1797.
- [10] Shukla JB, Parihar RS and Gupta SP. Effects of peripheral layer viscosity on blood flow through the artery with mild stenosis. *Bulletin of Mathematical Biology*. Vol 42, pp. 797-805. Pergamon Press Ltd. 1980.
- [11] Agarwal.S, Sinha A.K and Singh SP. A theoretical analysis of the effect of the non-newtonian bile flow on the flow characteristics in the diseased cystic duct. *Int. J. of Appl. Math and Mech*. 8(16): 92-103, 2012.
- [12] Agarwal.S, Sinha A.K and Singh SP. Effect of the plug flow on the flow characteristics of bile through diseased cystic duct: Casson Model Analysis *Advances in Applied Science Research*, 2012, 3 (2):1098-1106
- [13] Agarwal.S, Sinha A.K and Singh SP. Effect on the Flow of Bile Due to Radially Varying Viscosity in the Cystic Duct *International Journal of Scientific and Innovative Mathematical Research (IJSIMR)* Volume 2, Issue 2, March 2014, PP 180-185.